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Tracking Control of a Class of Hamiltonian Mechanical Systems with Disturbances

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Abstract

This paper presents a trajectory-tracking control strategy for a class of mechanical systems in Hamiltonian form. The class is characterised by a simplectic interconnection arising from the use of generalised coordinates and full actuation. The tracking error dynamic is modelled as a port-Hamiltonian Systems (PHS). The control action is designed to take the error dynamics into a desired closed-loop PHS characterised by a constant mass matrix and a potential energy with a minimum at the origin. A transformation of the momentum and a feedback control is exploited to obtain a constant generalised mass matrix in closed loop. The stability of the close-loop system is shown using the close-loop Hamiltonian as a Lyapunov function. The paper also considers the addition of integral action to design a robust controller that ensures tracking in spite of disturbances. As a case study, the proposed control design methodology is applied to a fully actuated robotic manipulator.

1 Introduction

Mechanical systems exhibit equilibrium points where the potential energy is stationary, and the stable equilibrium points are characterised by a minimum of the potential energy - this can be described using the Principle of Virtual Work [Greenwood, 2003]. When designing a motion control law for a mechanical system it is natural to exploit energy concepts and target a closed-loop energy with desirable characteristics such as having a minimum of the potential energy at the desired equilibrium point and the kinetic energy shaped to achieve certain dynamic behaviours. This has lead to formulations of control problems and solutions in terms of energy-based models such as Euler-Lagrange [Ortega *et al.*, 1998] and Hamiltonian [van der Schaft, 2006]. In this paper, we follow the latter.

We address the trajectory tracking control design problem for a class of mechanical systems with particular structure that arises from the use of generalised co-ordinates and full actuation. Many mechanical systems fall within this class—of particular importance are fully actuated robotic manipulators. The problem of trajectory tracking for these robotic manipulators has been addressed in term of Euler-Lagrange models by Ortega and Spong [1989] and by Wen and Bayard [1988]. Our formulation in terms of Hamiltonian models provides an alternative, which exploits the rich geometrical structure of the Hamiltonian formulation. In particular, we use a change of momentum co-ordinates and a feedback control that allow to set a constant generalised mass matrix in the target closed-loop system. This can be exploited for the tuning of the control law. We also consider the addition of integral action, for cases where constant disturbances must be rejected. We explote the procedure propose in [Donaire and Junco, 2009; Ortega and Romero, 2012; Romero *et al.*, 2013] for regulation problem, and we applied to the tracking problem. Finally, we illustrate the use of the proposed control design methodology on a fully-actuated robotic manipulator.

2 Mechanical Systems and Port Hamiltonian Models

The dynamics of mechanical systems, such as a robotic manipulator, can be described using the Euler-Lagrange equation [Lanczos, 1986]:

$$\frac{d}{dt}\left[\nabla_{\dot{q}} L(q,\dot{q})\right] - \nabla_q L(q,\dot{q}) = \tau, \tag{1}$$

where q and $\dot{q}$ are the n-dimensional vectors of generalised coordinates and velocities respectively, and τ is the vector of generalised forces.

The Lagrangian $L(q,\dot{q})$ is the difference between the kinetic co-energy and the potential energy of the system. For systems within the realm of classical mechanics, the

Lagrangian takes the form

$$L(q,\dot q) = T^*(q,\dot q) - V(q) = \frac{1}{2}\dot q^{\mathsf T} M(q)\dot q - V(q) \quad (2)$$

where the generalised mass matrix $M(q) > 0$ is symmetric for all q.

For these systems, the conjugate generalised momentum is $p = \nabla_{\dot q} L(q,\dot q) = M(q)\dot q$. Using the momentum and the generalised coordinate vector, the set of n second-order differential equations arising from (1) can be transformed, using the Legendre's transformation, into a set of $2n$ first-order differential equations [Lanczos, 1986]:

$$\dot q = \nabla_p H(p,q), \quad (3)$$
$$\dot p = -\nabla_q H(p,q) + \tau \quad (4)$$

where the Hamiltonian is the sum of the kinetic energy and the potential energy:

$$H(p,q) = T(q,p) + V(q) = \frac{1}{2} p^{\mathsf T} M^{-1}(q) p + V(q). \quad (5)$$

This function represents the total energy of the system. The equations (3)-(4) are called the Hamilton's canonical equations of motion.

In the control literature, the Hamiltonian model (3)-(4) has been generalised to what is known as a port-Hamiltonian system (PHS) (e.g. [van der Schaft, 2006]):

$$\begin{aligned} \dot x &= [J(x) - R(x)]\,\nabla H(x) + g(x)\,u, &(6)\\ y &= g^{\mathsf T}(x)\,\nabla H(x), &(7)\end{aligned}$$

where x is the state vector and the pair u, y are the input and output m-dimensional vectors. These are conjugate variables; that is, their inner product represents the power exchanged between the system and the environment. The function $J(x) = -J^{\mathsf T}(x)$ describes the power preserving interconnection structure through which the components of the system exchange energy. The symmetric function $R(x) \geq 0$ captures dissipative phenomena in the system. The function $g(x)$ weighs the action of the input on the system and defines the conjugate output. From (6)-(7), it follows that

$$\dot H(x) = y^{\mathsf T} u - [\nabla H(x)]^{\mathsf T} R(x) \nabla H(x) \leq y^{\mathsf T} u, \quad (8)$$

which shows passivity of the PHS model [van der Schaft, 2000].

In this paper, we consider fully-actuated mechanical systems with dynamics represented as follows

$$\begin{bmatrix}\dot q\\ \dot p\end{bmatrix} = \begin{bmatrix} 0 & I_n\\ -I_n & 0\end{bmatrix}\nabla H(q,p) + \begin{bmatrix}0\\ I_n\end{bmatrix}(\tau + \tau_d), \quad (9)$$

where $H(q,p)$ is given in (5), I_n is the $n\times n$ identity matrix, τ is the vector of control forces, and τ_d is the vector of disturbances forces. The disturbances can be decomposed as $\tau_d = \bar d + d(t)$, where $\bar d$ is a constant vector and $d(t)$ is a time-varying vector. In this model, the generalised force vector τ is mapped directly into the momentum equation due to the identity matrix. If the components of τ are independent, the system is within the class of fully actuated systems since $\dim\tau = \dim p = n$. For a definition of the complete class see Bullo and Lewis [2004]. Note also that this model uses generalised and not quasi-coordinates, which gives the simplectic form for the interconnection in the first term on the right-hand side of (9), namely,

$$J(x) = \begin{bmatrix} 0 & I_n\\ -I_n & 0\end{bmatrix}.$$

3 Tracking Control of Fully-actuated Mechanical Systems

We consider the dynamics of the system (9), and given a (possible) time-varying reference $q^*(t)$ and its time derivatives $\dot q^*(t)$ and $\ddot q^*(t)$, then the control problem is to design a controller that ensures

$$\lim_{t\to\infty} q(t) = q^*(t).$$

Proposition 3.1 *Consider the disturbance free ($\tau_d = 0$) port-Hamiltonian system (9) in closed loop with the control law*

$$\begin{aligned}\tau \;=\;& \nabla_q H(q,p) - M_d M^{-1}(q)\nabla\tilde V(\tilde q) + (J_2 - R_2)M_d^{-1}\tilde p +\\ +\;& M(q)\ddot q^* + \dot M(q)\dot q^* - K\nabla^2\tilde V(\tilde q)\Big[-M^{-1}(q)K\\ & \nabla\tilde V(\tilde q) + M^{-1}(q)\tilde p\Big], \end{aligned} \quad (10)$$

where

$$\begin{aligned}\tilde q &= q - q^*, &(11)\\ \tilde p &= p - M\dot q^* + K\nabla\tilde V, &(12)\end{aligned}$$

the $n\times n$ constant matrices K and M_d, and the $n\times n$ matrices J_2 and R_2 satisfy

$$M^{-1}K + K^{\mathsf T}M^{-1} > 0, \quad (13)$$
$$M_d = M_d^{\mathsf T} > 0, \quad (14)$$
$$R_2 = R_2^{\mathsf T} > 0, \quad (15)$$
$$J_2 = -J_2^{\mathsf T}, \quad (16)$$

and the function $\tilde V(\tilde q)$ has a (global) minimum at the origing, i.e. $\arg\min\tilde V(\tilde q) = 0$*. Then,*

i) **PHS form**. *The dynamics of the closed loop can be written as a PHS as follows*[1]

$$\begin{bmatrix}\dot{\tilde q}\\ \dot{\tilde p}\end{bmatrix} = \begin{bmatrix} -M^{-1}K & M^{-1}M_d\\ -M_d M^{-1} & J_2 - R_2\end{bmatrix}\nabla H_d(\tilde q,\tilde p), \quad (17)$$

[1] To simplify the notation, we drop the dependency of matrices and function on the independent variable, e.g. we note M and $\tilde V$ instead of $M(q)$ and $\tilde V(\tilde q)$.

with

$$H_d(\tilde{q},\tilde{p}) = \frac{1}{2}\tilde{p}^\mathsf{T} M_d^{-1}\tilde{p} + \tilde{V}(\tilde{q}) \quad (18)$$

ii) **Stability**. *The tracking error $\tilde{q}(t)$ asymptotically converges to zero, which ensures the control objective*

$$\lim_{t\to\infty} q(t) = q^*(t).$$

Proof We first consider the tracking error $\tilde{q}$ as in (11), and we compute the time derivative as follows

$$\begin{aligned}\dot{\tilde{q}} = \dot{q} - \dot{q}^* &= M^{-1}p - \dot{q}^* \\ &\equiv -M^{-1}K\nabla\tilde{V} + M^{-1}\tilde{p}. \quad (19)\end{aligned}$$

The last equality is satisfied for $\tilde{p}$ as in (12), which implies that the dynamics of $\tilde{q}$ can be written as in the first line in (17). Second, we compute the derivative of (12)

$$\begin{aligned}\dot{\tilde{p}} &= \dot{p} - M\ddot{q}^* - \dot{M}\dot{q}^* - K\nabla^2\tilde{V}\dot{\tilde{q}} \\ &= -\nabla_q H + \tau - M\ddot{q}^* - \dot{M}\dot{q}^* - K\nabla^2\tilde{V}[-M^{-1}K \\ &\quad \nabla\tilde{V} + M^{-1}\tilde{p}] \\ &\equiv -M_d M^{-1}\nabla\tilde{V} + (J_2 - R_2)M_d^{-1}\tilde{p}. \quad (20)\end{aligned}$$

It can be verified that the equality of the last two lines is satisfied by the control law (10). Therefore, the dynamics of $\tilde{p}$ can be written as in the second line of (17). Then, the dynamics of the closed loop in PHS form (17) follows directly from (19) and (20), which shows claim *i*).

To show tracking of the time-varying reference $q^*(t)$, we study the (global) asymptotic stability of the equilibrium $(\tilde{q}_\star,\tilde{p}_\star) = (0,0)$ of the close-loop dynamics (17). Since M_d is positive definite and symmetric, and the function $\tilde{V}$ has a (global) minimum at the origin, then the Hamiltonian H_d qualify as a Lyapunov candidate function. The derivative of H_d along the solutions of (17) is as follows

$$\begin{aligned}\dot{H}_d(\tilde{q},\tilde{p}) &= \begin{bmatrix} (\nabla_{\tilde{q}}H_d)^\mathsf{T} & (\nabla_{\tilde{p}}H_d)^\mathsf{T}\end{bmatrix}\begin{bmatrix}\dot{\tilde{q}} \\ \dot{\tilde{p}}\end{bmatrix} = \\ &= -(\nabla_{\tilde{q}}H_d)^\mathsf{T} M^{-1}K\nabla_{\tilde{q}}H_d + (\nabla_{\tilde{q}}H_d)^\mathsf{T} M^{-1} \\ &\quad M_d\nabla_{\tilde{p}}H_d - (\nabla_{\tilde{p}}H_d)^\mathsf{T} M_d M^{-1}\nabla_{\tilde{q}}H_d + \\ &\quad +(\nabla_{\tilde{p}}H_d)^\mathsf{T}(J_2 - R_2)\nabla_{\tilde{p}}H_d \\ &= -\frac{1}{2}(\nabla_{\tilde{q}}\tilde{V})^\mathsf{T}(M^{-1}K + K^\mathsf{T}M^{-1})\nabla_{\tilde{q}}\tilde{V} - \\ &\quad -\tilde{p}^\mathsf{T} M_d^{-\mathsf{T}} R_2 M_d^{-1}\tilde{p} \\ &< 0 \quad (21)\end{aligned}$$

where we use the properties (13)-(16). Since the time derivative of the Lyapunov function is negative semidefinite, then the equilibrium $(\tilde{q}_\star,\tilde{p}_\star)$ is (almost globally) asymptotically stable. This fact implies that $\tilde{q}(t) \to zero$ as time goes to infinite, therefore $\lim_{t\to\infty} q(t) = q^*(t)$, which ensures the tracking of the reference. □□□

In the following proposition, we consider the robust tracking problem of a mechanical system (9) for a given a time-varying reference $q^*(t)$ (and its time derivatives $\dot{q}^*(t)$ and $\ddot{q}^*(t)$). The control controller ensures

$$\lim_{t\to\infty} q(t) = q^*(t),$$

in spite of unknown constant disturbance τ_d. In case of bounded disturbances $d(t)$, the controller proposed ensures input-to-state-stability (ISS). This property has been used in control theory and control design (see [Sontag, 2008] for a recent survey). Two nice robust property of ISS systems are that trajectories of the states are bounded if the inputs are bounded, and the states are convergent if the inputs are convergent. These two properties are easily attained for linear systems, but it can be more difficult to obtain for nonlinear systems.

Proposition 3.2 *Consider port-Hamiltonian system (9) in closed loop with the control law*[2]

$$\begin{aligned}\tau &= \nabla_q H(p,q) - M_d M^{-1}(q)\nabla\tilde{V}(\tilde{q}) + (J_2 - R_2)M_d^{-1} \\ &\quad \Big[p - M(q)\dot{q}^* + K\nabla\tilde{V}(\tilde{q}) - M_d K_z z\Big] - (J_3 - R_3)^\mathsf{T} \\ &\quad K_z z + M(q)\ddot{q}^* + \dot{M}(q)\dot{q}^* + M_d K_z \dot{z} - \\ &\quad -K\nabla^2\tilde{V}(\tilde{q})\Big[M^{-1}(q)p + \dot{q}^*\Big], \quad (22)\end{aligned}$$

where

$$\begin{aligned}\dot{z} &= -\left[M_d M^{-1}(q) + (J_3 - R_3)M_d K\right]\nabla\tilde{V}(\tilde{q}) + \\ &\quad +(J_3 - R_3)M_d^{-1}M(q)\left[M^{-1}(q)p - \dot{q}^*\right], \quad (23)\end{aligned}$$

and

$$\tilde{q} = q - q^*, \quad (24)$$

The $n\times n$ constant matrices K and M_d, and the $n\times n$ matrices J_2 and R_2 satisfy

$$\begin{aligned} M^{-1}(q)K + K^\mathsf{T}M^{-1}(q) &> \epsilon I_n, & (25)\\ M_d = M_d^\mathsf{T} &> 0, & (26)\\ K_z = K_z^\mathsf{T} &> 0, & (27)\\ R_2 = R_2^\mathsf{T} &> 0, & (28)\\ R_3 = R_3^\mathsf{T} &> 0, & (29)\\ J_2 &= -J_2^\mathsf{T}, & (30)\\ J_3 &= -J_3^\mathsf{T}, & (31)\end{aligned}$$

with $\epsilon \in \mathbb{R}_{>0}$, and the function $\tilde{V}(\tilde{q})$ has a global minimum at the origin, i.e. $\arg\min\tilde{V}(\tilde{q}) = 0$*, and*

$$\begin{aligned} c_1|\tilde{q}|^2 \le \tilde{V}(\tilde{q}) &\le c_2|\tilde{q}|^2, & (32)\\ c_4|\tilde{q}|^2 \le \left|\nabla\tilde{V}(\tilde{q})\right|^2 &\le c_3|\tilde{q}|^2, & (33)\end{aligned}$$

[2]Note that the control law (22)-(23) requires no information about the value of the disturbance.

with $c_1, c_2, c_3, c_4 \in \mathbb{R}_{>0}$. *Then,*

i) **PHS form**. *The dynamics of the closed loop can be written as a PHS as follows*

$$\begin{bmatrix} \dot{\tilde{q}} \\ \dot{\tilde{p}} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -M^{-1}K & M^{-1}M_d & M^{-1}M_d \\ -M_d M^{-1} & J_2 - R_2 & -(J_3 - R_3)^\top \\ -M_d M^{-1} & J_3 - R_3 & J_3 - R_3 \end{bmatrix} \nabla W + \\ + \begin{bmatrix} 0 \\ d(t) \\ 0 \end{bmatrix} \tag{34}$$

with

$$W(\tilde{q}, \tilde{p}, z) = \frac{1}{2}\tilde{p}^\top M_d^{-1}\tilde{p} + \tilde{V}(\tilde{q}) + (z - \alpha)^\top K_z (z - \alpha), \tag{35}$$

and

$$\alpha = K_z^{-1}(J_2 - R_2 - J_3 - R_3)^{-1}\bar{d}$$

ii) **Robust Exponential Stability**. *For the case of constant unknown disturbances* $\bar{d} \neq 0$ *and time-varying disturbance* $d(t) = 0$, *the controller* (22)-(23) *ensures that the tracking error* $\tilde{q}(t)$ *exponentially converges to zero, therefore the control objetive is achieved, i.e.*

$$\lim_{t \to \infty} q(t) = q^*(t)$$

ii) **Input-to-State-Stability**. *For the case of constant unknown disturbances* $\bar{d} \neq 0$ *and time-varying disturbance* $d(t) \neq 0$, *the controller* (22)-(23) *ensures input-to-state-stability of the system with input* $d(t)$.

Proof We first consider the tracking error $\tilde{q}$ as in (24), and we compute the time derivative as follows

$$\begin{aligned} \dot{\tilde{q}} &= \dot{q} - \dot{q}^* = M^{-1}p - \dot{q}^* \\ &\equiv -M^{-1}K\nabla\tilde{V} + M^{-1}\tilde{p} + M_d M^{-1}K_z(z - \alpha), \end{aligned} \tag{36}$$

which is satisfied if

$$\tilde{p} = p - M\dot{q}^* + K\nabla\tilde{V} - M_d K_z(z - \alpha), \tag{37}$$

therefor the dynamics of $\tilde{q}$ can be written as in the first line in (34). Second, we compute the derivative of (37)

$$\begin{aligned} \dot{\tilde{p}} &= \dot{p} - M\ddot{q}^* - \dot{M}\dot{q}^* - K\nabla^2\tilde{V}\dot{\tilde{q}} - M_d K_z \dot{z} \\ &= -\nabla_q H + \tau + \bar{d} + d(t) - M\ddot{q}^* - \dot{M}\dot{q}^* - K\nabla^2\tilde{V} \\ &\quad [\dot{q} - \dot{q}^*] - M_d K_z \dot{z} \\ &\equiv -M_d M^{-1}\nabla\tilde{V} + (J_2 - R_2)M_d^{-1}\tilde{p} - (J_3 - R_3)^\top \\ &\quad K_z(z - \alpha) + d(t) \end{aligned} \tag{38}$$

It can be verified that the equality of the last two lines is satisfied by the control law (22). Therefore, the dynamics of $\tilde{p}$ can be written as in the second line of (34). Then, the dynamics of the closed loop in PHS form (34) follows directly from (36) and (38), which shows claim *i*).

To show exponential tracking of the time-varying reference $q^*(t)$ claimed in *ii*), we study the global exponential stability of the equilibrium $(\tilde{q}_\star, \tilde{p}_\star, z_\star) = (0, 0, \alpha)$ of the close-loop dynamics (34). Since M_d is positive definite and symmetric, and the function $\tilde{V}$ has a global minimum at the origin and satisfies (32), then the Hamiltonian W qualify as a Lyapunov candidate function, and it also satisfies

$$\beta_1 |(\tilde{q}, \tilde{p}, z - \alpha)|^2 \leq W(\tilde{q}, \tilde{p}, z) \leq \beta_2 |(\tilde{q}, \tilde{p}, z - \alpha)|^2 \tag{39}$$

with $\beta_1, \beta_2 \in \mathbb{R}_{>0}$. We compute the derivative of W along the solutions of (34) as follows

$$\begin{aligned} \dot{W}_d(\tilde{q}, \tilde{p}, z) &= \begin{bmatrix} (\nabla_{\tilde{q}}W)^\top & (\nabla_{\tilde{p}}W)^\top & (\nabla_z W)^\top \end{bmatrix} \begin{bmatrix} \dot{\tilde{q}} \\ \dot{\tilde{p}} \\ \dot{z} \end{bmatrix} = \\ &= -(\nabla_{\tilde{q}}W)^\top M^{-1}K\nabla_{\tilde{q}}W + (\nabla_{\tilde{q}}W)^\top M^{-1} \\ &\quad M_d\nabla_{\tilde{p}}W + (\nabla_{\tilde{q}}W)^\top M^{-1}M_d\nabla_z W - \\ &\quad -(\nabla_{\tilde{p}}W)^\top M_d M^{-1}\nabla_{\tilde{q}}W + (\nabla_{\tilde{p}}W)^\top \\ &\quad (J_2 - R_2)\nabla_{\tilde{p}}W - (\nabla_{\tilde{p}}W)^\top(J_3 - R_3)^\top \\ &\quad \nabla_z W - (\nabla_z W)^\top M_d M^{-1}\nabla_{\tilde{q}}W + \\ &\quad +(\nabla_z W)^\top(J_3 - R_3)\nabla_{\tilde{p}}W + (\nabla_z W)^\top \\ &\quad (J_3 - R_3)\nabla_z W + (\nabla_{\tilde{p}}W)^\top d(t) \\ &= -\frac{1}{2}(\nabla_{\tilde{q}}\tilde{V})^\top(M^{-1}K + K^\top M^{-1})\nabla_{\tilde{q}}\tilde{V} - \\ &\quad -(\nabla_{\tilde{p}}W)^\top R_2\nabla_{\tilde{p}}W - (z - \alpha)^\top K_z R_3 K_z \\ &\quad (z - \alpha) + (\nabla_{\tilde{p}}W)^\top d(t) \\ &= -\frac{\epsilon}{2}(\nabla_{\tilde{q}}\tilde{V})^\top\nabla_{\tilde{q}}\tilde{V} - \lambda_1|(z - \alpha)|^2 - \\ &\quad -(\nabla_{\tilde{p}}W)^\top R_2\nabla_{\tilde{p}}W + (\nabla_{\tilde{p}}W)^\top d(t) \\ &\leq -\frac{\epsilon}{2}\left|\nabla_{\tilde{q}}\tilde{V}\right|^2 - \lambda_1|(z - \alpha)|^2 - \\ &\quad -\frac{\lambda_2}{2}\left|\nabla_{\tilde{p}}W\right|^2 + \frac{1}{2\lambda_2}\left|d(t)\right|^2 \\ &\leq -\frac{\epsilon c_3}{2}\left|\tilde{q}\right|^2 - \frac{\lambda_2\lambda_3}{2}\left|\tilde{p}\right|^2 - \lambda_1|(z - \alpha)|^2 \\ &\quad +\frac{1}{2\lambda_2}\left|d(t)\right|^2 \\ &\leq -\frac{\gamma}{\beta_2}W(\tilde{q}, \tilde{p}, z) + \frac{1}{2\lambda_2}\left|d(t)\right|^2, \end{aligned} \tag{40}$$

where $\lambda_1 = \lambda_{min}(K_z R_3 K_z)$, $\lambda_2 = \lambda_{min}(R_2)$, $\lambda_3 = \lambda_{min}(M_d^{-1})$, and $\gamma = \min\{\epsilon c_3, \lambda_2\lambda_3, 2\lambda_1\}$. The operator $\lambda_{min}(A)$ computes the minimum eigenvalue of the matriz A. To probe exponential stability of the closed loop with constant disturbances, we set $d(t) = 0$ in (40). Then the inequality

$$\dot{W}_d(\tilde{q}, \tilde{p}, z) \leq -\frac{\gamma}{\beta_2}W(\tilde{q}, \tilde{p}, z)$$

together with (39) ensure (global) exponential stability of the equilibrium $(\tilde{q}_\star, \tilde{p}_\star, z_\star) = (0, 0, \alpha)$ (see e.g. [Khalil, 2000]). This fact implies that $\tilde{q}(t)$ exponential converge to zero as time goes to infinite, therefore $\lim_{t\to\infty} q(t)$ exponentially converge to $q^*(t)$, in spite of the presence of unknown constant disturbances. The input-to-state-stability property follows considering $d(t) \neq 0$ in (40) as follows

$$\dot{W}_d(\tilde{q}, \tilde{p}, z) \leq -\frac{\gamma}{\beta_2} W(\tilde{q}, \tilde{p}, z) + \frac{1}{2\lambda_2} |d(t)|^2 ,$$

which show W is an ISS-Lyapunov function ([Sontag, 2008; Khalil, 2000]). □□□

4 Case of Study

In this section, we present simulation results of a robotic manipulator to evaluate the performance of the control design. We consider a two-link 2 DoF serial-robotic manipulator shown in Figure 1. The generalised coordinate vector is $q = (q_1, q_2)$. The mass matrix is

$$M(q) = \begin{bmatrix} m_1 l_{c1}^2 + m_2 l_1^2 & m_2 l_{c1} l_1 cos(q_1 - q_2) \\ m_2 l_{c1} l_1 cos(q_1 - q_2) & m_2 l_{c2}^2 \end{bmatrix} \quad (41)$$

where the parameters l_{ci}, m_i are the offset to the centre of mass and the mass of the link i respectively, and l_1 is the length of the link 1. The potential energy is

$$V(q) = (m_1 c_1 + m_2 l_1) g \sin(q_1) + m_2 c_2 g \sin(q_2). \quad (42)$$

To ensure stability, we choose the desired potential energy on the position error as follows

$$\tilde{V}(\tilde{q}) = \frac{1}{2} \tilde{q}^\mathsf{T} Q \tilde{q}, \quad (43)$$

where $Q > 0$ is a diagonal constant matrix. Note that since $V(\tilde{q})$ is a quadratic function, then it has a global minimum at the origin, $\nabla\tilde{V}(\tilde{q}) = Q\tilde{q}$ and $\nabla^2\tilde{V}(\tilde{q}) = Q$. In this example, the matrices M_d, R_2, R_3 and K_z are constant diagonal matrices, $K = kI_n$ with $k \in \mathbb{R}_{>0}$, and $J_2 = J_3 = 0$.

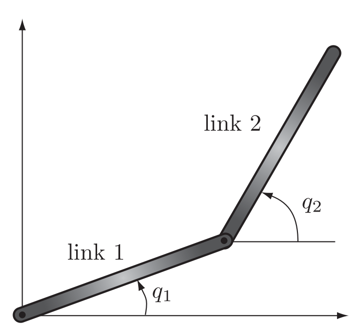

Figure 1: Two-link robotic manipulator

The reference vector is $q^* = (q_1^*, q_2^*)$, where $q_1^*(t)$ and $q_2^*(t)$ are two time-varying, smooth and sinusoidal-type signals provided to the controller. The simulation starts with the two links in the horizontal position, with two constant disturbances acting on link 1 at t=5s and on link 2 at t=10s. In this scenario, we test the control designs from propositions 3.1 and 3.2, identified here as controller 1 and controller 2.

Figures 2, 6, 4 and 8 display the angle and angular velocity references together with the current angle and angular velocities of the two links for controller 1 and controller 2. It can be seen from these plots that the control law tracks the reference throughout the entire simulation, with the disturbances affecting the angles and angular velocities of the manipulator for only a small interval of time. Although very small, there is a small tracking error in the controller 1 due to the disturbance as expected (see figure 3). The controller 2 compensates the constant disturbance and the tracking error converge to zero (see figure 7).

Figure 5 and 9 show the computed torques supplied by the controller 1 and 2 to the robotic manipulator, respectively. As can be seen, the control torque are smooth and within acceptable bounds. Finally, Figure 10 shown the time history of the state of the controller 2, namely the integral action.

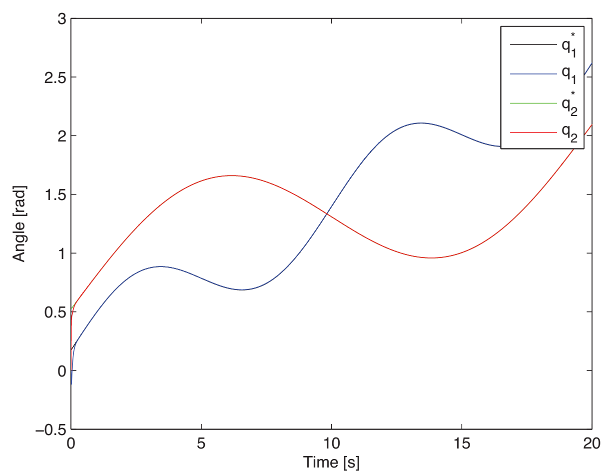

Figure 2: Link angles and its references (controller 1).

5 Conclusions

We present a control design for general mechanical systems that ensures tracking of time-varying references. The first controller shown in proposition 3.1 ensures asymptotic tracking for time-varying references. The second controller, presented in proposition 3.2, is augmented with integral action, which ensures exponential convergency of the tracking error to zero in spite of con-

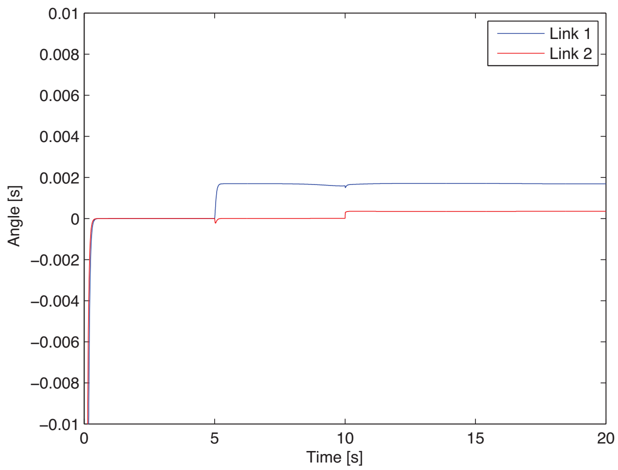

Figure 3: Angle errors $\tilde{q} = q - q^*$ (controller 1).

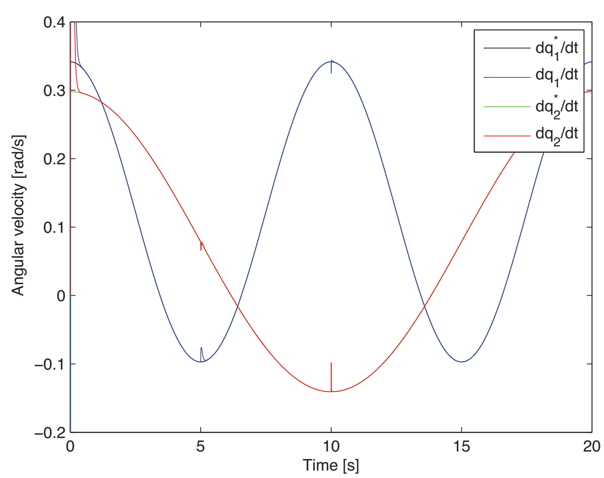

Figure 4: Link velocities and its references (controller 1).

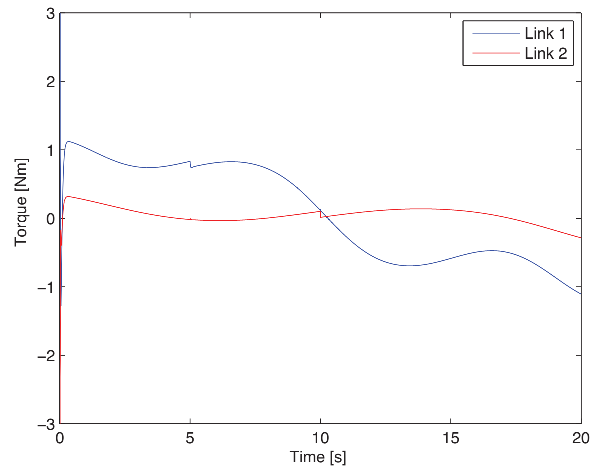

Figure 5: Control torques (controller 1).

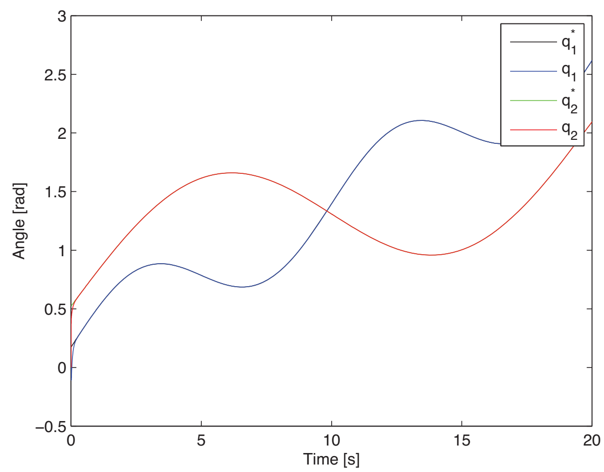

Figure 6: Link angles and its references (controller 2).

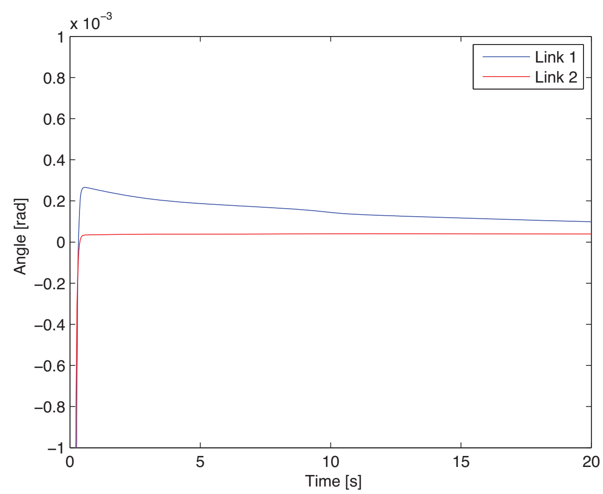

Figure 7: Angle errors $\tilde{q} = q - q^*$ (controller 2).

stant disturbances. Also, the robustness of the controller with respect to disturbances is studied in term of input-to-state-stability theory. Using the proposed design, the stability analysis of the closed-loop system does not require the use of Barbalat's lemma. Indeed, since both close-loop systems (17) and (34) have damping in all the states, the Hamiltonians H_d and W are strict Lyapunov functions for the closed loop dynamics. The performance of the controllers are illustrated in simulation with a robotic manipulator. The control design proposed in this paper requires shaping the form of both potential and kinetic energy functions. Our current research focus on control designs that shape only the potential energy, whilst preserving the form of the open-loop mass matrix.

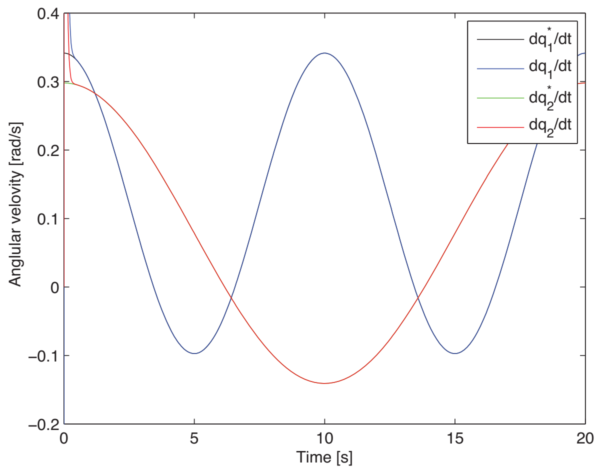

Figure 8: Link velocities and its references (controller 2).

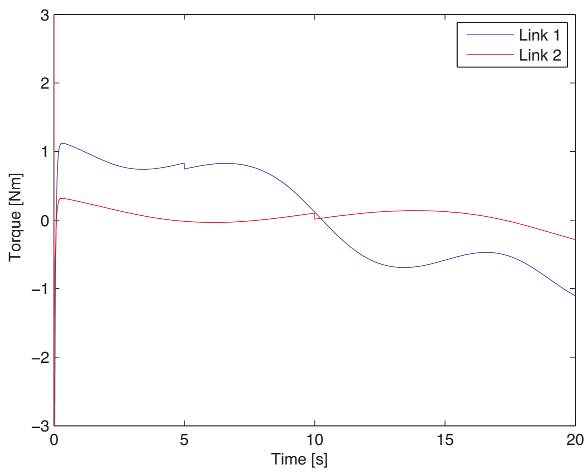

Figure 9: Control torques (controller 2).

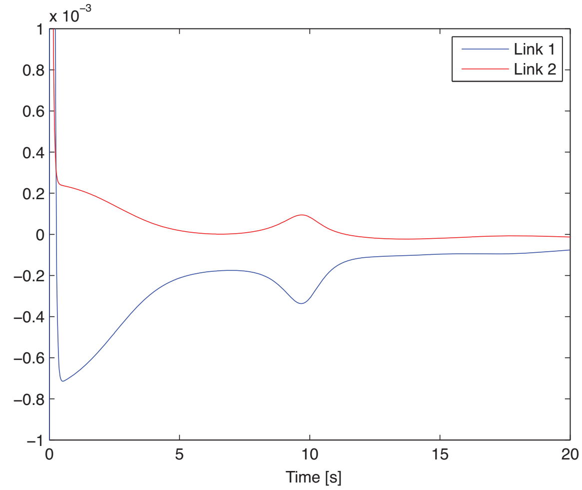

Figure 10: State of the integral action control (controller 2).

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