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An Open-Source Implementation of a Unit Quaternion based Attitude and Trajectory Tracking for Quadrotors

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Abstract

In this paper, we present results on the implementation of a hierarchical quaternion based attitude and trajectory controller for manual and autonomous flights of quadrotors. Unlike previous papers on using quaternion representation, we use the non-linear complementary filter that estimates the attitude in quaternions and as such does not involve Euler angles or rotation matrices. We show that for precise trajectory tracking, the resulting attitude error dynamics of the system is non-autonomous and is almost globally asymptotically and locally exponentially stable under the proposed control law. We also show local exponential stability of the translational dynamics under the proposed trajectory tracking controller which sits at the highest level of the hierarchy. Thus by input-to-state stability, the entire system is locally exponentially stable. The quaternion based observer and controllers are available as open-source.

1 INTRODUCTION

Quadrotors are aerial vehicles with four rotors for generating lift and controllability. Their light weight, ease of design and simple dynamics have made them the de facto standard for aerial vehicle studies and has led to the development and implementation of many control algorithms and open-source projects [Lim *et al.*, 2012]. In recent years, quadrotors have been made to do many impressive aggressive maneuvers and tasks. Some of these maneuvers include the grasping and flights through narrow openings [Mellinger *et al.*, 2012], multiple flips [Lupashin *et al.*, 2010] and single stall turns [Huang *et al.*, 2009]. The majority of these papers are based on using the concept of hierarchical control wherein the control problem is divided into position, attitude and motor control [Hua *et al.*, 2013; Bangura and Mahony, 2014].

For attitude representation, the majority of the papers have used Euler angles [Castillo *et al.*, 2004; Salazar-Cruz *et al.*, 2005; Bouabdallah *et al.*, 2004]. Although the Euler angle representation is intuitive, it has several disadvantages including the fact that it suffers from gimbal lock (*i.e.* singularity problem) and the estimation process is embedded in a 3×3 rotation matrix. To overcome the singularity problem of Euler angles in aggressive maneuvers, rotation matrices to control attitude was used in [Mellinger and Kumar, 2011]. In [Lee *et al.*, 2010], a Lyapunov based non-linear controller formulated on $\mathrm{SE}(3)$ was proposed. By defining an attitude and an angular velocity error function, the authors showed exponential stability of the attitude and position error dynamics of their proposed geometric controller. An extension of this work which included feedforward terms in angular velocity was implemented on a quadrotor in [Mellinger and Kumar, 2011]. However, the majority of the implementation of rotation matrix based controllers are carried out on ground station computers. This is due to constraints associated with matrix representation such as storage and computations (including normalisation to reduce rounding off errors) which may not be available on a heavily loaded embedded computer.

Quaternions are the standard attitude representation in three-dimensional computer graphic engines. It is due to the fact that they do not suffer from gimbal lock and consume less memory for computations such as normalisation and suffer less from rounding errors than rotation matrices. Hence one of the major applications of quaternion based attitude observers and controllers will be in the full automation of miniature insect like micro-robots [Ozcan *et al.*, 2013] and cheap low-cost quadrotors with limited computational capability. [Tayebi and McGilvray, 2006] implemented the first set of quaternion based attitude stabilisation controller for a four-rotor fixed in space quadrotor. They showed through Lyapunov analysis that their proposed PD attitude controller is globally asymptotically stable. However, they obtained the attitude quaternions from converting the Euler angles of the attitude. This prevented the implementation to suffer from the double cover problem associated with quaternions. To overcome this problem, [Mayhew *et al.*, 2013]

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proposed a hybrid system framework that keeps a memory of the error quaternion. [Monte and Lohmann, 2013; Zhao *et al.*, 2013] used the backstepping approach to decouple the translational and yaw dynamics to do a full quaternion controller. In 2013, [Diao *et al.*, 2013] developed extensive theory with Lyapunov stability proofs on height and attitude control using quaternions. [Wang and Yu, 2010] designed a regulator for the SE(3) version of quaternions, the dual quaternion. However, all of these have failed to show a practical implementation on quadrotors which can open its use to the quadrotor community. To the authors' best knowledge, none of the existing work using quaternion control deals with trajectory tracking and stability analysis of the resulting non-autonomous error system.

In this paper, we present experimental results and implementation of a quaternion based attitude and trajectory tracking controller on an onboard microcontroller for manual and autonomous control of quadrotors. Using the non-linear complementary filter, we estimate the attitude in quaternions that exhibit the known double cover property associated with such a representation [Mahony *et al.*, 2008]. We extend the current attitude and trajectory tracking controllers to include feedforward terms for precise trajectory tracking. The controllers are designed to exploit the natural passivity of the system using Lyapunov stability analysis. The proposed trajectory controller is locally exponentially stable and does not involve feedback linearisation of the dynamics about an equilibrium point. We use the dynamic properties of the vehicle to algebraically determine additional feedforward terms (angular velocity and acceleration) for the attitude controller. With these feedforward terms, we prove using Lyapunov analysis an almost global asymptotic and local exponential stability of the non-autonomous attitude error dynamics in tracking the given trajectory. We illustrate our controllers with experimental results and show usage of the controllers in manual flight using an RC transmitter as well as autonomous position control and tracking of a trajectory. The different controller codes implemented for the PX4 autopilot and a guide to tuning the different gains are made available as open-source [1].

The remainder of the paper is organised as follows: the non-linear quadrotor model is presented in Section 2; the attitude and trajectory controllers in Section 3 and 4 respectively. Finally details on the implementation and experimental results are shown in Section 5.

2 QUADROTOR NON-LINEAR MODEL

Consider the quadrotor shown in Fig. 1 with mass m and moment of inertia $I \in \mathbb{R}^{3\times3}$. The relative position of the body frame $\{B\}$ to the inertial frame $\{A\}$ is denoted by $\zeta = (x,y,z)^\top \in \{A\}$, and $v = (v_x, v_y, v_z)^\top \in \{A\}$ denotes the linear velocity of $\{B\}$ with respect to $\{A\}$ expressed in $\{A\}$. If the orientation of $\{B\}$ with respect to $\{A\}$ is $R = {}^A R_B \in \mathrm{SO}(3)$

[1]Code repository on github: `http://goo.gl/ypw6Tx`

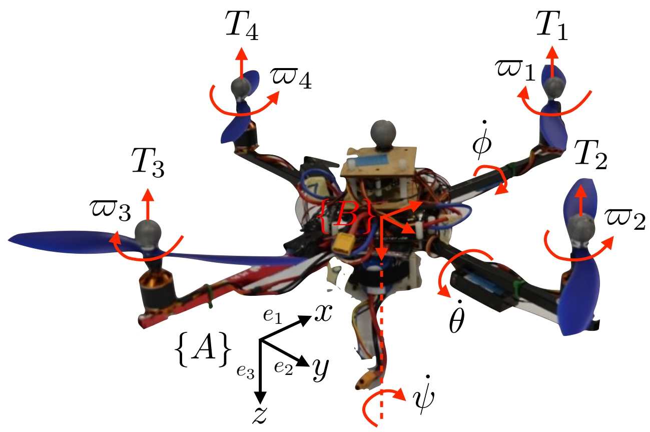

Figure 1: The quadrotor used in this paper. The figure also shows the different frames of reference, thrust forces and rotor rotation directions.

and the angular velocity of $\{B\}$ with respect to $\{A\}$ is defined by $\Omega = (\Omega_x, \Omega_y, \Omega_z)^\top \in \{B\}$. Finally if we let $T \in \mathbb{R}, \tau \in \mathbb{R}^3$ be the combined thrust force and torque generated by the rotors respectively, then the non-linear model for a quadrotor is given by [Bangura and Mahony, 2012]

$$\dot\zeta = v, \tag{1a}$$
$$m\dot v = mge_3 - TRe_3, \tag{1b}$$
$$\dot R = RS(\Omega), \tag{1c}$$
$$I\dot\Omega = -S(\Omega)I\Omega + G_a(\Omega) + \tau, \tag{1d}$$

where $S(\Omega) \in \mathbb{R}^{3\times3}$ is the skew symmetric matrix derived from $\Omega \in \mathbb{R}^3$ such that $S(\Omega)w = \Omega \times w$ for all $w \in \mathbb{R}^3$ and e_3 denotes the unit vector in z direction. The gyroscopic effect $G_a(\Omega)$ in (1d) of the rotors is given by

$$G_a(\Omega) := -\sum_{i=1}^{4}(-1)^i \varpi_i I_r(\Omega \times e_3), \tag{2}$$

where ϖ_i is the angular velocity of rotor i, $I_r \in \mathbb{R}$ is the moment of inertia of a rotor blade and the factor $(-1)^i$ accounts for direction of rotation of the given rotor. The total thrust magnitude, T on the vehicle is the sum of the individual thrusts T_i of the rotors and is defined by

$$T = \sum_{i=1}^{4} T_i.$$

The required thrust for the individual motors is determined using the following relationship [Hamel *et al.*, 2002]

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ l & 0 & -l & 0 \\ 0 & -l & 0 & l \\ \kappa & -\kappa & \kappa & -\kappa \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} T \\ \tau_x \\ \tau_y \\ \tau_z \end{bmatrix}, \tag{3}$$

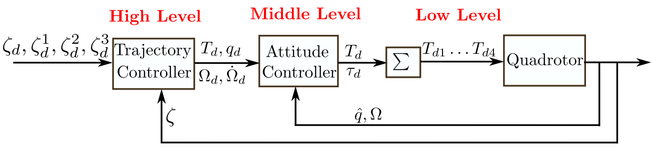

Figure 2: Proposed hierarchical control structure.

where l is the length of each rotor from the centre of $\{B\}$ and κ is the thrust to torque ratio of the rotor blades.

Our proposed hierarchical control architecture is shown in Fig. 2. With a desired trajectory and vehicle position fed to the trajectory tracking controller (high level), the outputs are the desired setpoints for the attitude controller (middle level) which outputs the torques required. The torque and total thrust T are then converted to individual motor thrust forces using (3) and used as setpoints for the low level motor controllers. The high level trajectory controller runs at $20Hz$ with response times in the order of seconds. The middle level attitude controllers run at $250Hz$ and have response times around a few hundred milliseconds. The low level controllers use a static model for thrust (T) to rotor speed (ϖ)

$$T = C_{T0}\varpi + C_T\varpi^2,$$

where C_{T0} and C_T are thrust constants determined experimentally. The rotor speed is used as a setpoint in achieving the desired thrust. In controlling ϖ, a high gain feedback K_ϖ is used along with a feedforward voltage calibrated to match steady state ϖ. Measurements at this level occur at $1kHz$ and the rise time of ϖ is under $50ms$ [Bangura and Mahony, 2014]. Hence there is a natural time scale separation between the different dynamic levels of the control architecture.

3 QUADROTOR ATTITUDE CONTROL

In this section, we present the quaternion based attitude controller for precise trajectory tracking. We start with the non-linear complementary filter for attitude estimation. We then present the attitude controller for tracking bounded trajectories with quaternion, angular velocity and acceleration set-points. Unlike [Tayebi and McGilvray, 2006], we show that the error dynamics of the attitude are non-autonomous and that they are almost globally asymptotically and locally exponentially stable under the attitude controller tracking these setpoints that are specified by a robust trajectory controller.

3.1 Quaternion theory

If the attitude of a quadrotor given in Euler angles is (ϕ, θ, ψ), then the attitude in rotation matrix R and unit quaternion $q = (q_0, \vec{q})^\top$, where $q_0 \in \mathbb{R}$, $\vec{q} = (q_1,\ q_2,\ q_3)^\top \in \mathbb{R}^3$,$|q| = 1$ is given by (4) and (5) defined using the $'roll - pitch - yaw'$ rotation convention

$$R = \begin{pmatrix} c\theta c\psi & s\phi s\theta c\psi - c\phi s\psi & c\phi s\theta c\psi + s\phi s\psi \\ c\theta s\psi & s\phi s\theta s\psi + c\phi c\psi & c\phi s\theta s\psi - s\phi c\psi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{pmatrix}, \quad (4)$$

$$q = \begin{bmatrix} c\frac{\psi}{2}c\frac{\theta}{2}c\frac{\phi}{2} + s\frac{\psi}{2}s\frac{\theta}{2}s\frac{\phi}{2} \\ c\frac{\psi}{2}c\frac{\theta}{2}s\frac{\phi}{2} - s\frac{\psi}{2}s\frac{\theta}{2}c\frac{\phi}{2} \\ c\frac{\psi}{2}s\frac{\theta}{2}c\frac{\phi}{2} + s\frac{\psi}{2}c\frac{\theta}{2}s\frac{\phi}{2} \\ -c\frac{\psi}{2}s\frac{\theta}{2}s\frac{\phi}{2} + s\frac{\psi}{2}c\frac{\theta}{2}c\frac{\phi}{2} \end{bmatrix}, \quad (5)$$

where c and s represent the cos and sin of an angle respectively. For quaternion to Euler by (6)

$$\begin{aligned} \phi &= \tan^{-1}\left(\frac{2(q_0q_1 + q_2q_3)}{1 - 2q_1q_1 + q_2q_2}\right), \quad (6) \\ \theta &= \sin^{-1}\left(2(q_0q_2 - q_3q_1)\right), \\ \psi &= \tan^{-1}\left(\frac{2(q_0q_3 + q_1q_2)}{1 - 2(q_2q_2 + q_3q_3)}\right). \end{aligned}$$

Using the Rodrigues formula, R is obtained from q by

$$R(q) = (q_0^2 - \vec{q}^\top\vec{q})I_{3\times3} + 2\vec{q}\vec{q}^\top - 2q_0S(\vec{q}). \quad (7)$$

Definitions for detailed quaternion algebra can be found in references such as [Tayebi and McGilvray, 2006]. The associated kinematic representation of (1c) in quaternion form is given by

$$\begin{aligned} \dot{\vec{q}} &= \frac{1}{2}\left(S(\vec{q}) + q_0\mathbb{I}_{3\times3}\right)\Omega, \\ \dot{q}_0 &= -\frac{1}{2}\Omega^\top\vec{q}. \quad (8) \end{aligned}$$

This can also be written as

$$\dot{q} = \frac{1}{2}q \otimes \mathfrak{p}(\Omega), \quad (9)$$

where $\otimes$ is the quaternion multiplication and $\mathfrak{p}(\Omega) = (0, \Omega)^\top$ is angular velocity expressed in the quaternion algebra.

3.2 Non-linear Complementary Filter

Unlike [Tayebi and McGilvray, 2006] which used a linear complementary filter to estimate the attitude in Euler angles and converted them to quaternions, our implementation uses the explicit complementary filter (ECF) proposed in [Mahony *et al.*, 2008] to estimate the attitude in quaternions.

The ECF filter uses vectorial measurements from 9-DoF sensors consisting of accelerometer, gyroscope and magnetometer. Let the predicted measurements of the i-th sensor that we align with the body frame be denoted by $\hat{v}_i$ and v_i the actual normalised sensor measurement in body frame. In our implementation, we use magnetometer $v_{i,mag}$, accelerometer $v_{i,acc}$ and gyroscope for Ω. The attitude observer in quaternion is defined as follows

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \otimes p\left(\Omega - \hat{b} - k_P\sum_{i=1}^{n} k_i(\hat{v}_i \times v_i)\right), \quad (10)$$

$$\dot{\hat{b}} = k_I\sum_{i=1}^{n} k_i(\hat{v}_i \times v_i), \quad (11)$$

$$\hat{v}_i = p^\dagger\left(\hat{q}^{-1} \otimes \mathfrak{p}(\hat{v}_i) \otimes \hat{q}\right), \quad (12)$$

where the $\mathring{v}_i \in \mathbb{R}^3$ is a predefined reference direction of the field in the inertial frame, $\hat{b} \in \mathbb{R}^3$ is the estimated bias, $k_P, k_I \in \mathbb{R}^{3\times3}$ are diagonal gain matrices, $k_i, k_p \in \mathbb{R}^{3\times3}$ are the mixing sensitivity diagonal gain matrices and $p^\dagger$ is an operator that extracts the vector part of the quaternion (*i.e.,* an inverse operation of $\mathfrak{p}(\cdot)$).

To estimate $\hat{v}_i$ of each sensor, we need a $\mathring{v}_i$ which is a predefined reference direction in $\{A\}$. The gravitational direction $\mathring{v}_{i,grav} = [0,\ 0,\ 1]^\top$ is always fixed in inertial frame though the magnetometer is not. The Earth's magnetic field can be considered as $\mathring{v}_{i,mag} = [b_x,\ 0,\ b_z]^\top$ to have components in horizontal and vertical axis.

The predicted gravitational direction in $\{B\}$ from the estimated quaternion is defined as follows

$$\hat{v}_{i,acc} = p^\dagger\left(\hat{q}^{-1} \otimes \mathfrak{p}(\mathring{v}_i) \otimes \hat{q}\right) = \begin{bmatrix} 2(\hat{q}_1\hat{q}_3 - \hat{q}_0\hat{q}_2) \\ 2(\hat{q}_0\hat{q}_1 + \hat{q}_2\hat{q}_3) \\ \hat{q}_0^2 - \hat{q}_1^2 - \hat{q}_2^2 + \hat{q}_3^2 \end{bmatrix}. \quad (13)$$

The Earth's magnetic field in $\{B\}$ is

$$\hat{v}_{i,mag} = \begin{bmatrix} 2b_x(0.6 - q_2^2 - q_3^2) + 2b_z(q_1q_3 - q_0q_2) \\ 2b_x(q_1q_2 - q_0q_3) + 2b_z(q_0q_1 + q_2q_3) \\ 2b_x(q_0q_2 + q_1q_3) + 2b_z(0.5 - q_1^2 - q_3^2) \end{bmatrix}. \quad (14)$$

However, the $\mathring{v}_{i,mag}$ should also be estimated from $v_{i,mag}$ as it is not a fixed quantity (*i.e.,* it depends on the location of the IMU on the Earth). Let $h = [h_x,\ h_y,\ h_z]^\top$ be the magnetometer readings in $\{A\}$ and raw magnetometer reading $v_{i,mag} = [m_x,\ m_y,\ m_z]^\top$, then h can be computed as follows

$$h = p^\dagger\left(\hat{q} \otimes \mathfrak{p}(v_{i,mag}) \otimes \hat{q}^{-1}\right) = \begin{bmatrix} 2\Big(m_x(0.5 - q_2^2 - q_3^2) + m_y(q_1q_2 - q_0q_3) + m_z(q_1q_3 + q_0q_2)\Big) \\ 2\Big(m_x(q_1q_2 + q_0q_3) + m_y(0.5 - q_1^2 - q_3^2) + m_z(q_2q_3 - q_0q_1)\Big) \\ 2\Big(m_x(q_1q_3 - q_0q_2) + m_y(q_2q_3 + q_0q_1) + m_z(0.5 - q_1^2 - q_2^2)\Big) \end{bmatrix}. \quad (15)$$

$\mathring{v}_{i,mag}$ is defined as

$$\mathring{v}_{i,mag} = \left[\sqrt{h_x^2 + h_y^2},\ 0,\ h_z\right]. \quad (16)$$

For the purpose of control, we make the following assumption.

Assumption 1. *The attitude observer response is such that the attitude of the vehicle $q = \hat{q}$.*

Since it is important that the attitude filter be allowed to stabilise prior to flight implies that the transient error convergence of the filter is not of consequence in the closed loop response. The all pass nature of the complementary filter along with the high sample rate of the filter make Assumption 1 a viable assumption in practice.

3.3 Attitude Control For Precise Trajectory Tracking

If q is the current attitude of the vehicle and q_d is the desired attitude, then the error quaternion we define by

$$q_e = q_d^{-1} \otimes q. \quad (17)$$

The angular velocity error we also define by

$$\tilde{\Omega} = \Omega - \Omega_d, \quad (18)$$

where Ω_d is the desired angular velocity. Taking the derivative of (17) and carrying out some algebraic simplifications, one obtains the error kinematics

$$\dot{q}_e = -\frac{1}{2}\mathfrak{p}(\Omega_d) \otimes q_e + \frac{1}{2} q_e \otimes \mathfrak{p}(\Omega). \quad (19)$$

Let

$$\mathbb{P}(q) : \Re^4 \to \Re,$$

denote the projection such that

$$\mathbb{P}(q_0, \vec{q}) := q_0. \quad (20)$$

It is easily verified that for any quaternion q and angular velocity $\mathfrak{p}(\Omega)$

$$\mathbb{P}(q \otimes \mathfrak{p}(\Omega)) = \mathbb{P}(\mathfrak{p}(\Omega) \otimes q) = -\vec{q}^\top\Omega = -\Omega^\top\vec{q}.$$

Applying the projection operator $\mathbb{P}$ on (19), we get

$$\dot{q}_{e_0} = -\frac{1}{2}\tilde{\Omega}^\top\vec{q}_e. \quad (21)$$

Let $\dot{\Omega}_d$ denote the desired angular acceleration of the desired trajectory, we will consider a feedforward torque defined by

$$\tau_d := I\dot{\Omega}_d + S(\Omega_d)I\Omega - G_a(\Omega_d). \quad (22)$$

The error dynamics are given by

$$\begin{aligned} I\dot{\tilde{\Omega}} &= I\dot{\Omega} - I\dot{\Omega}_d, \\ I\dot{\tilde{\Omega}} &= -S(\Omega)I\Omega + G_a(\Omega) + \tau - I\dot{\Omega}_d. \end{aligned} \quad (23)$$

Theorem 1. *Let Ω_d denote a desired trajectory for the attitude dynamics and assume that $\dot{\Omega}_d$ is known and that the signals Ω_d, $\dot{\Omega}_d$ and $\ddot{\Omega}_d$ exist and are bounded. Let $k > 0$ be a scalar gain and let $K_q, K_\Omega \in \mathbb{R}^{3\times3}$ be positive definite symmetric gain matrices such that $\mathbb{I} - \frac{1}{k}K_\Omega K_q = 0$. Consider the control law*

$$\tau = \tau_d - K_\Omega\left(\tilde{\Omega} + K_q\vec{q}_e\right). \quad (24)$$

Then the error kinematics (19) *and error dynamics* (23) *of the attitude are almost globally asymptotically and locally exponentially stable around the identity quaternion $q_I = (1,0,0,0)^\top$ and angular rate error $\tilde{\Omega} = (0,0,0)^\top$ under the control law* (24).

Proof. Substituting for τ in (23), we obtain the non-autonomous error dynamics

$$I\dot{\tilde{\Omega}} = -S(\tilde{\Omega})I\Omega + G_a(\tilde{\Omega}) - K_\Omega\left(\tilde{\Omega} + K_q\vec{q}_e\right). \qquad (25)$$

Consider the following Lyapunov function

$$V_1\left(q_e,\tilde{\Omega}\right) = 2\left(1-q_{e_0}\right) + \frac{1}{2k}\tilde{\Omega}^\top I\tilde{\Omega}. \qquad (26)$$

The derivative of V_1 is

$$\frac{\mathrm{d}}{\mathrm{dt}}V_1 = -2\dot{q}_{e_0} + \frac{1}{k}\tilde{\Omega}^\top I\dot{\tilde{\Omega}}. \qquad (27)$$

Substituting for (21) and (23), (27) becomes

$$\dot{V}_1 = \tilde{\Omega}^\top\vec{q}_e + \frac{1}{k}\tilde{\Omega}^\top\left(-S(\tilde{\Omega})I\Omega + G_a(\tilde{\Omega}) - K_\Omega\left(\tilde{\Omega}+K_q\vec{q}_e\right)\right). \qquad (28)$$

Since $\tilde{\Omega}^\top S(\tilde{\Omega}) = \tilde{\Omega}\times\tilde{\Omega} = 0$, we are left with

$$\dot{V}_1 = \tilde{\Omega}^\top\vec{q}_e + \frac{1}{k}\tilde{\Omega}^\top\left(-K_\Omega\tilde{\Omega} - K_\Omega K_q\vec{q}_e\right). \qquad (29)$$

Therefore

$$\dot{V}_1 = -\frac{1}{k}\tilde{\Omega}^\top K_\Omega\tilde{\Omega} + \tilde{\Omega}^\top\vec{q}_e - \frac{1}{k}\tilde{\Omega}^\top K_\Omega K_q\vec{q}_e, \qquad (30)$$

Hence from the theorem statement,

$$\dot{V}_1 = -\frac{1}{k}\tilde{\Omega}^\top K_\Omega\tilde{\Omega}. \qquad (31)$$

Since V_1 is positive definite and unbounded in $\tilde{\Omega}$, while the quaternion group is compact, then $\tilde{\Omega}$ and q_e are bounded signals and since the reference signal Ω_d is also bounded, it follows that solutions exist for all time and all signals are bounded.

The derivative of $\dot{V}_1(t)$ is

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2}V_1 = -\frac{2}{k}\tilde{\Omega}^\top K_\Omega I^{-1}\left(-S(\tilde{\Omega})I\Omega + G_a(\tilde{\Omega}) - K_\Omega(\tilde{\Omega}+K_q\vec{q}_e)\right). \qquad (32)$$

All signals on the right-hand side of (32) are bounded and this demonstrates that $\dot{V}_1$ is uniformly continuous. Applying Barbalat's lemma, it follows that $\dot{V}_1$ converges to zero and hence $\tilde{\Omega}$ is globally asymptotically stable to zero. To show that $q_e \to q_I$, it is first required to show that $\dot{\tilde{\Omega}} \to 0$ as $t\to\infty$. To do this, we will use Barbalat's lemma a second time. Observe that the integral

$$\left|\int_0^T \frac{d}{dt}||I\tilde{\Omega}||dt\right| = \left|||I\tilde{\Omega}(T)|| - ||I\tilde{\Omega}(0)||\right| < \infty,$$

is bounded since $\tilde{\Omega}$ is globally asymptotically stable. It is straightforward to verify that $\frac{d^2}{dt^2}||I\tilde{\Omega}||$ is bounded by using boundedness of the system signals, in particular, boundedness of $\ddot{\Omega}_d$. It follows that $\frac{d}{dt}\|I\tilde{\Omega}\|$ is uniformly continuous and Barbalat's lemma shows that $\frac{d}{dt}\|I\tilde{\Omega}\| \to 0$. It is straightforward to verify that this implies that $\|I\dot{\tilde{\Omega}}\| \to 0$.

Consider the error dynamics (25), rearranging and taking norms yields

$$\begin{aligned}\|K_\Omega K_q\vec{q}_e\| &= \|I\dot{\tilde{\Omega}} + S(\tilde{\Omega})I\Omega - G_a(\tilde{\Omega}) + K_\Omega\tilde{\Omega}\|,\\ &\le \|I\dot{\tilde{\Omega}}\| + \|S(\tilde{\Omega})I\Omega\| + \|G_a(\tilde{\Omega})\| + \|K_\Omega\tilde{\Omega}\|.\end{aligned}$$

The terms on the right hand side are all globally asymptotically stable to zero and it follows that $\|K_\Omega K_q\vec{q}_e\| \to 0$. Since $K_\Omega K_q$ is full rank, it follows that $q_e \to \pm q_I$. To show almost global asymptotic stability of $(q_I, 0)$, we show that by suitable choice of $k > 0$ we can ensure that the basin of attraction of $(q_I, 0)$ is as large as desired, excluding the set $(-q_I, \tilde{\Omega}(t_0))$, a set of measure zero.

For any initial condition $(q_e(t_0), \tilde{\Omega}(t_0))$ such that $q_{e_0} \neq -1$, there exists $k > 0$ such that

$$V_1(q_e(t_0),\tilde{\Omega}(t_0)) = 2(1-q_{e_0}(t_0)) + \frac{1}{2k}\tilde{\Omega}(t_0)^\top I\tilde{\Omega}(t_0) < 4.$$

It follows that $q_e(t) \not\to -q_I$ since

$$\begin{aligned}&V_1(q_e(t),\tilde{\Omega}(t)) < V_1(q_e(t_0),\tilde{\Omega}(t_0))\\ &\quad < V_1(-q_I,0) \le V_1(-q_I,\tilde{\Omega}(t_0)),\end{aligned}$$

and hence for this choice of gain $k > 0$, the initial condition $(q_e(t_0), \tilde{\Omega})$ must lie in the basin of attraction of $(q_I, 0)$. Therefore the non-autonomous error dynamics (23) are almost semi-globally asymptotically stable under the control law (24).

To prove local exponential stability, reconsider (19) which can be rewritten as

$$\dot{q}_e = -\frac{1}{2}\mathfrak{p}\left(\Omega_d\right)\otimes q_e + \frac{1}{2}q_e\otimes\mathfrak{p}\left(\Omega - \Omega_d + \Omega_d\right), \qquad (33)$$

$$\dot{q}_e = \frac{1}{2}\left[q_e, \mathfrak{p}\left(\Omega_d\right)\right] + \frac{1}{2}q_e\otimes\mathfrak{p}\left(\tilde{\Omega}\right), \qquad (34)$$

where $[A,B] = A\otimes B - B\otimes A$ is the quaternion commutator. If z_q and z_Ω are the linear approximations for q_e and $\tilde{\Omega}$ around q_I and $\tilde{\Omega} = 0$ such that $q_e \approx q_I + \mathfrak{p}(z_q)$ and $z_\Omega \approx \tilde{\Omega}$ with $z_q, z_\Omega \in \mathbb{R}^3$. The new linear system $z = (z_q, z_\Omega)^\top$ based on the error kinematics (34) and dynamics (25) is given by

$$\begin{aligned}\begin{pmatrix}\dot{z}_q\\ \dot{z}_\Omega\end{pmatrix} = &\left[\begin{pmatrix}S(\Omega_d) & 0\\ 0 & -I^{-1}S(I\Omega) + m_\Omega\end{pmatrix}\right.\\ &\left. + \begin{pmatrix}0 & \frac{1}{2}I_{3\times3}\\ -K_qI^{-1} & -K_qK_\Omega I^{-1}\end{pmatrix}\right]\begin{pmatrix}z_q\\ z_\Omega\end{pmatrix},\end{aligned} \qquad (35)$$

where $m_\Omega = \sum_{i=1}^4(-1)^i\varpi_i I_r S(e_3)$. Equation (35) can be written as $\dot{z} = A(t)z$ where the time varying part of the matrix $A(t)$ is as a result of Ω_d and Ω and $\forall t, A(t)$ is bounded. If we choose the Lyapunov function $V_l = z_q^\top K_q z_q + z_\Omega^\top I z_\Omega$ with $\dot{V}_l = -z_\Omega K_q K_\Omega z_\Omega$ and let $C = (0, I_3) \in \mathbb{R}^{6\times6}$, by looking at

the state transition matrix and the observability Gramian, it can be shown that $(A(t), C)$ is Uniformly Continuously Observable which implies asymptotic stability [Khalil, 1996; Thienel and Sanner, 2003]. For linear systems, asymptotic stability implies exponential stability which proves local exponential stability. □

4 QUADROTOR TRAJECTORY CONTROL

In this section, we present the trajectory tracking controller which outputs the desired setpoints for the attitude controller. We show using Lyapunov analysis that the translational error dynamics are locally exponentially stable under the proposed control law. In this way, we exploit the natural passivity of the system to design a robust trajectory tracking controller. The novelty of our approach is the use of the derivatives of both the vehicle position and desired trajectory in an algebraic manner to determine the desired attitude setpoint and feedforward terms for the attitude controller (24) of Section 3.

From the previous sections, we can make the following necessary assumptions

Assumption 2. *1. The individual desired thrust forces equal the output thrusts i.e. $T_d(i) = T(i), i = 1, \ldots, 4$.*

2. The attitude of the vehicle equals the desired attitude i.e. $q_d = q$.

4.1 Trajectory Controller

From Section 2, if the position $\zeta = (x, y, z)^\top$, the desired position $\zeta_d = (x_d, y_d, z_d)^\top$ and $\tilde{\zeta} = \zeta - \zeta_d$. From (1b), the error dynamics for the translational states is given by

$$m\ddot{\tilde{\zeta}} = mge_3 - T_d Re_3 - m\ddot{\zeta}_d. \tag{36}$$

Remark 1. *It should be noted that the feedforward acceleration $\ddot{\zeta}_d$ should contain ge_3 given the manner in which accelerometers for quadrotors work. This already appears in* (36). *As such, $\ddot{\zeta}_d$ does not contain ge_3.*

Theorem 2. *Let $\dot{\zeta}_d$ and $\ddot{\zeta}_d$ of a desired trajectory be known and are such that $\tilde{\zeta}$ and $\dot{\tilde{\zeta}}$ are bounded. Let $K_p, K_d \in \mathbb{R}^{3\times 3}$ be diagonal positive definite gain matrices. Consider the trajectory tracking under the control law*

$$T_d R^d e_3 = mge_3 + K_p\tilde{\zeta} + K_d\dot{\tilde{\zeta}} - m\ddot{\zeta}_d. \tag{37}$$

Then the error dynamics of (36) *are locally asymptotically stable around the equilibrium point $\tilde{\zeta} = \dot{\tilde{\zeta}} = (0,0,0)^\top$.*

Proof. From Assumption 2 and (36), the new error dynamics under (37) is thus

$$m\ddot{\tilde{\zeta}} = -K_p\tilde{\zeta} - K_d\dot{\tilde{\zeta}}. \tag{38}$$

This is an autonomous block diagonal linear system in error coordinates with Eigenvalues which are asymptotically and therefore globally exponentially stable.

Remark 2. *In practice because the assumptions won't hold, this should only be relied upon for local stability. Therefore, system* (36) *is locally exponentially stable under the control law of* (37).

Hence by the notion of input-to-state stability, given that the middle and top level controllers are both locally exponentially stable, the entire system is locally exponentially stable around $\tilde{\zeta} = \dot{\tilde{\zeta}} = 0$ [Khalil, 1996, Lemma 5.6]. □

4.2 Attitude Controller Setpoint Determination

It is now left to show the derivation of the setpoints in q, Ω and $\dot{\Omega}$ for the controller in Section 3. For simplicity of notation and based on Assumption 2, throughout this subsection, $q = q_d = (q_0, \vec{q})^\top$, where $\vec{q} = (q_1, q_2, q_3)^\top$. Since the translational dynamics equation (1b) is written in rotation matrices, we write the desired attitude as $R(q_d)$ using Rodrigues formula (7). The desired attitude in e_3 is given by

$$R(q_d)e_3 = \begin{pmatrix} 0 \\ 0 \\ q_0^2 - \vec{q}^\top\vec{q} \end{pmatrix} + 2\begin{pmatrix} q_3q_1 \\ q_3q_2 \\ q_3q_3 \end{pmatrix} - 2q_0\begin{pmatrix} q_2 \\ -q_1 \\ 0 \end{pmatrix}. \tag{39}$$

Remark 3. *For quadrotors, the orientation about the e_3 direction ψ can be set as desired. In a similar way, we set $q_3 = 0$. It should be noted that this does not imply that $\psi(t) = 0\ \forall t$. Hence, Ω_{d_z} along with its higher derivatives can be set to zero or based on the desired trajectory of q_3.*

Simplifying (39) based on Remark 3,

$$R(q_d)e_3 = \begin{pmatrix} -2q_0q_2 \\ 2q_0q_1 \\ q_0^2 - (q_1^2 + q_2^2) \end{pmatrix}. \tag{40}$$

If the terms on the right hand side of (37) are x_{con}, y_{con} (the orientation of the thrust vector) and z_{con} for x, y and z respectively i.e.

$$R(q_d)e_3 = \begin{pmatrix} x_{con} \\ y_{con} \\ z_{con} \end{pmatrix},$$

are given by

$$x_{con} = \frac{\left(mge_3 + K_p\tilde{\zeta} + K_d\dot{\tilde{\zeta}} - m\ddot{\zeta}_d\right)^\top e_1}{T_d}, \tag{41}$$

$$y_{con} = \frac{\left(mge_3 + K_p\tilde{\zeta} + K_d\dot{\tilde{\zeta}} - m\ddot{\zeta}_d\right)^\top e_2}{T_d}, \tag{42}$$

$$z_{con} = \left(mge_3 + K_p\tilde{\zeta} + K_d\dot{\tilde{\zeta}} - m\ddot{\zeta}_d\right)^\top e_3. \tag{43}$$

From (40), one can also obtain the following

$$T_d = \frac{z_{con}}{q_0^2 - \vec{q}^\top\vec{q}}, \tag{44}$$

$$x_{con} = -2q_0q_2, \tag{45}$$

$$y_{con} = 2q_0q_1. \tag{46}$$

With $q_3 = 0$ and $\|q\| = 1$ implies

$$q_0^2 = \frac{1 \pm \sqrt{1-(x_{con}^2 + y_{con}^2)}}{2}.$$

We choose

$$q_0 = \pm\sqrt{\frac{1 + \sqrt{1-(x_{con}^2 + y_{con}^2)}}{2}}. \tag{47}$$

Substituting q_0 into (44) to (46), we obtain the desired thrust T_d and attitude in quaternion q_d. In Section 5, we will show how the sign of q_0 is chosen to ensure that we always avoid the double cover problem associated with quaternions *i.e.* q_{e_0} converging to -1. Looking at Equations (44) to (47), one can see that the controller fails at the following conditions

$$q_0^2 = \vec{q}^\top \vec{q} = \frac{1}{\sqrt{2}}, \tag{48}$$

$$q_0 = 0, \tag{49}$$

$$x_{con}^2 + y_{con}^2 > 1. \tag{50}$$

Looking at the first condition, if $q_3 = 0$, implies that we get a set of attitudes in Euler angles that is $(\phi, \theta, \psi) = (90, 0, 0)^\circ$, $(0, 90, 0)$ and a range of other angles for which $\psi >>> 0$. These incorrectly represent the desired orientation and θ or $\phi = 90$ represent extreme maneuvres and a loss of control. From (47), $q_0 \neq 0$, hence, the condition of (49) is always met. Condition (50) is met by carefully choosing the gains K_p and K_d for the translational dynamics based on the bounds of $\|\tilde{\zeta}\|$, $\|\dot{\tilde{\zeta}}\|$ and the range of T_d.

Remark 4. *For the purpose of controller gains tuning, since this is usually first done for hovering, it is necessary to set* $T_d = 1$ *in* (41) *and* (42)*. Thereafter the values of the gains* K_p *and* K_d *scaled based on* T_d*.*

Unlike previous trajectory controllers such as [Mellinger and Kumar, 2011], as have been pointed out, our control system does not use feedback linearisation. For robustness of tracking, we assume availability of the derivatives of ζ and ζ_d required to determine additional terms for precise tracking. Consider taking the derivative of (37),

$$T_d \dot{R}(q_d) e_3 + \dot{T}_d R(q_d) e_3 = Kp \dot{\tilde{\zeta}} + K_d \ddot{\tilde{\zeta}} - m \dddot{\zeta}_d. \tag{51}$$

We know from (1c)

$$\dot{R}(q_d) = R(q_d) S(\Omega_d).$$

Thus

$$T_d R(q_d) S(\Omega_d) e_3 + \dot{T}_d R(q_d) e_3 = Kp \dot{\tilde{\zeta}} + K_d \ddot{\tilde{\zeta}} - m \dddot{\zeta}_d.$$

Since T_d and $R(q_d)$ have been found from (37), we substitute them and determine $\Omega_{d_x}, \Omega_{d_y}$ and $\dot{T}_d$. To obtain $\dot{\Omega}_{d_x}$ and $\dot{\Omega}_{d_y}$, we differentiate further to use the jerk ($\dddot{\zeta}$). Higher derivatives, $\dddot{\zeta}_d$ and beyond can be ignored as they may not be available from the desired trajectory.

$$T_d \left(\dot{R}(q_d) S(\Omega_d) e_3 + R(q_d) S(\dot{\Omega}_d) e_3\right) + \dot{T}_d R(q_d) S(\Omega_d) e_3 + \ddot{T}_d R(q_d) e_3 + \dot{T}_d \dot{R}(q_d) e_3 = Kp \ddot{\tilde{\zeta}} + K_d \dddot{\tilde{\zeta}}.$$

Substituting for $\dot{R}(q_d)$

$$T_d \left(\dot{R}(q_d) S(\Omega_d) e_3 + R(q_d) S(\dot{\Omega}_d) e_3\right) + 2\dot{T}_d R(q_d) S(\Omega_d) e_3 + \ddot{T}_d R(q_d) e_3 = Kp \ddot{\tilde{\zeta}} + K_d \dddot{\tilde{\zeta}}. \tag{52}$$

Therefore

$$T_d R(q_d) S(\dot{\Omega}_d) e_3 + \ddot{T}_d R(q_d) e_3 = -T_d R(q_d) S(\Omega_d) S(\Omega_d) e_3 \tag{53}$$

$$-2\dot{T}_d R(q_d) S(\Omega_d) e_3 + Kp \ddot{\tilde{\zeta}} + K_d \dddot{\tilde{\zeta}}, \tag{54}$$

which is solved for $\dot{\Omega}_{d_x}, \dot{\Omega}_{d_y}$ and $\ddot{T}_d$. Though $\ddot{T}_d$ is never used, it gives an indication of the feasibility of the trajectory and is upper-bounded by the motor transient response. With q_d, T_d, Ω_d and $\dot{\Omega}_d$ determined algebraically, they are then used as the setpoints for the attitude controller of Section 3.

5 IMPLEMENTATION AND RESULTS

In this section, we outline the implementation and present experimental results on the use of the attitude and trajectory controllers presented in Section 3 and 4.

5.1 Implementation Hardware

Both the trajectory tracking and attitude estimator and controllers are implemented on the PX4 open-source autopilot [Meier *et al.*, 2011]. The PX4 is equipped with MPU6000 and HMC5883L for measuring accelerations, rotation rates and the Earth's magnetic field. These sensor readings are used in the non-linear complementary filter for attitude estimation. Both the attitude estimator and controller have a higher thread priority and run at 250Hz. Due to communication delays, data losses and thread prioritisations, the trajectory controller runs at an average of 20Hz while the I2C communication between the autopilot and speed controllers is at 50Hz. Connected to the autopilot is an RC receiver for controlling the vehicle in manual mode and for mode switching between autonomous and manual control modes. The total weight of the quadrotor platform is 1.2kg with an arm length of $l = 50cm$. By setting $k = 1$, K_Ω and K_q can then be tuned.

5.2 Manual Flight

This usually involves manually controlling the attitude and height of the vehicle. To ensure uniformity with the primary PX4 attitude controller, the desired inputs from the pilot are T, Ω_z, ϕ and θ. In order to make the controller usable by

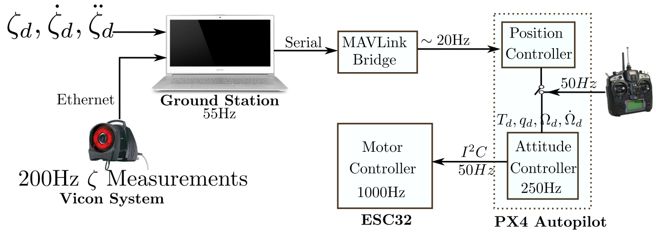

Figure 3: Layout of the implementation showing the various components of the control architecture. Due to delays in the serial communication link and data packet losses, the position along with desired trajectory data limits the trajectory controller to run at 20Hz though the attitude controller runs at 250Hz. On the vehicle, we do velocity estimation and use the onboard sensors for the remaining data.

the hobby community and knowing that quaternions may not have any physical meaning and the fact that manual mode involves controlling Ω_z and not q_3, we convert the attitude quaternions into Euler angles using (6) and then convert to quaternions with $\psi = 0$. In a similar manner, the pilot's inputs which are the desired attitude $(\phi^d, \theta^d, 0)^\top$ are also converted to quaternions using (5). Since manual flight mode does not involve tracking trajectories, the desired feedforward terms $(\Omega_x, \Omega_y, \dot{\Omega}_x, \dot{\Omega}_y, \dot{\Omega}_z)$ are set to zero. This mode always has $q_{e_0} \geq 0$.

5.3 Autonomous Flight

Our proposed architecture for autonomous control is shown in Fig.3. We use VICON system for determination of the position of the vehicle. This position along with the desired position or trajectory and its derivatives are sent to the quadrotor using a 3DR 915MHz radio. All the other data are obtained from sensors on the PX4. For every set of Euler angles, there are two sets of quaternions. Externally controlling the wrong quaternion will result in an unwanted rotation of 2π of the attitude of the vehicle. To overcome this problem, we choose the sign of q_{d_0} from the trajectory controller of (47) to match that of the q_0 from the attitude observer during every initialisation of the controller. This ensures that $q_{e_0} \geq 0$.

5.4 Experimental Results

The experimental results for control of different positions and a trajectory are shown in Fig. 4. The trajectory is a "figure 8" that is used without determining the reachability of the trajectory through close examination of $\ddot{T}_d$ using (53) and the motor rise time. The "figure 8" is defined by

$$x = 0.6\cos\left(\frac{\pi}{10}t\right), y = 0.6\sin\left(2\frac{\pi}{10}t\right), z = -1.0,$$

$$\dot{x} = -0.06\pi\sin\left(\frac{\pi}{10}t\right), \dot{y} = 0.12\pi\cos\left(2\frac{\pi}{10}t\right), \dot{z} = 0,$$

$$\ddot{x} = -0.6\left(\frac{\pi}{10}\right)^2\cos\left(\frac{\pi}{10}t\right), \ddot{y} = -2.4\left(\frac{\pi}{10}\right)^2\sin\left(2\frac{\pi}{10}t\right),$$

$$\ddot{z} = 0,$$

where t is the time since the start of the trajectory.

6 CONCLUSIONS

In this paper, we have presented theory, implementation and experimental results of a quaternion based attitude and trajectory tracking controller. We have extended the currently available quaternion based attitude stabilisation controllers to track both the attitude and its angular velocity and acceleration as feedforward terms to improve precise tracking of trajectories. In our proposed trajectory controller, we have shown how the controller can produce these feedforward terms algebraically. A Lyapunov stability analysis was carried out to prove an almost global asymptotic and local exponential stability of the resulting non-autonomous error dynamics under the attitude controller and local exponential stability for the trajectory controller. Unlike previous literature, we have implemented the observer and controllers on an embedded autopilot and make available the code as open-source.

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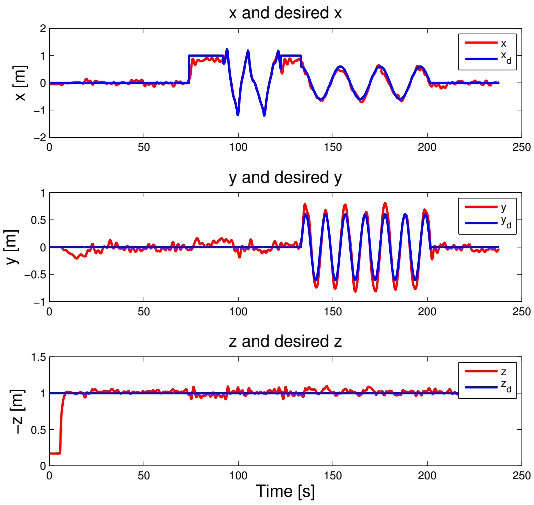
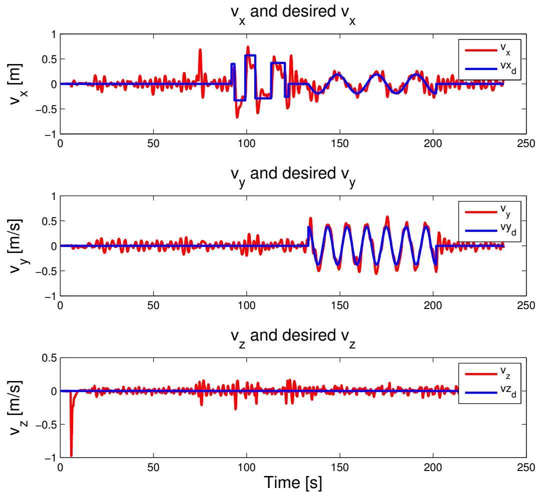
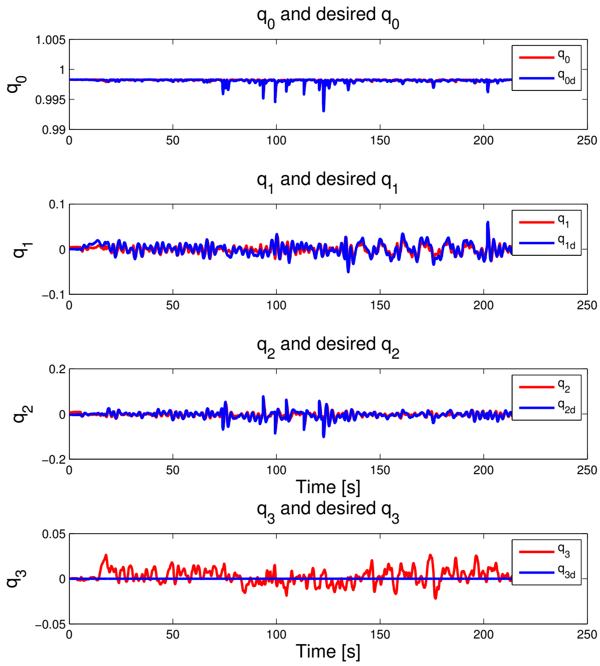


(a) Position (ζ) and desired position (ζ_d).



(b) Velocity ($\dot{\zeta}$) and desired velocity ($\dot{\zeta}_d$).



(c) Attitude (q)and desired attitude in quaternions (q_d).

Figure 4: Results for the quadrotor taking off, translational position, velocity control, trajectory tracking and hover recorded onboard the autopilot. The low frequency data received on the PX4 had a profound influence on the performance of the controller mainly on the 50% low-pass velocity estimation.

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