A coupled estimation and control analysis for attitude stabilisation of mini aerial vehicles.

Robert Mahony,
Dep. Eng., ANU, ACT,
0200, Australia
Robert.Mahony@anu.edu.au

Sung-Han Cha,
Dep. Eng., ANU, ACT,
0200, Australia
u3951656@anu.edu.au

Tarek Hamel,
I3S-CNRS, Nice-Sophia
Antipolis, France
thamel@i3s.unice.fr

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Abstract

Commercially viable aerial robotic vehicles require robust and efficient, but low-cost attitude (vehicle orientation) stabilisation systems. A typical attitude stabilisation system employs a low-cost IMU and consists of an attitude estimator as well as an attitude controller. This paper proposes a coupled non-linear attitude estimation and control design for the attitude stabilisation of low-cost aerial robotic vehicles. Attitude estimation is based on a non-linear complementary filter expressed on the rotation group. The attitude control algorithm is based on a non-linear control Lyapunov function analysis derived directly in terms of the rigid-body attitude dynamics. The interaction terms are bounded in terms of estimation and control errors and the full coupled system is shown to be (almost) globally stable.

1 Introduction

The last decade has seen an intense world wide effort in the development of mini aerial vehicles (mAV). Such vehicles are characterised by; small scale (dimensions of the order of 60cm), limited payload capacity, and embedded avionics systems (Office of Secretary of Defence, 2005). A subset of these mAV systems are the class of vertical take off and landing (VTOL) systems. A key component of the avionics system in a mAV is the attitude sensing and stabilisation control subsystem. Such systems must be highly reliable and have low computational overhead to avoid overloading the limited computational resources available in some applications. Traditional linear and extended Kalman filter techniques (Jun *et al.* , 1999; Bachmann *et al.* , 2001; Rehbinder & Hu, 2004), and more sophisticated filter techniques that include models of the system (Bryson & Sukkarieh, 2004), suffer from issues associated with poor modelling of the system (in particular, characterisation of noise within the system necessary for tuning filter parameters) as well as potentially high computational requirements. In recent work by the authors (Mahony *et al.* , 2005; Hamel & Mahony, 2006) a complementary filter has been proposed that provides a computationally cheap implementation of a simple and robust attitude estimation scheme that fully respects the non-linearities of rigid-body motion. This work is closely related to work that uses the quaternion formulation for the design of non-linear attitude filters (Salcudean, 1991; Vik & Fossen, 2001; Thienel & Sanner, 2003). In other recent work (Tayebi & McGilvray, 2004; Tayebi & McGilvray, 2006) a fully non-linear control algorithm for dynamic stabilisation of VTOL mAV systems has been developed based on the quaternion formulation for rigid body dynamics. Early work in this area predates the development of quaternion based filters (B. Wie & Arapostathis, 1989; Wen & Kreutz-Delgado, 1991; Fjellstad & I.Fossen, 1994). To the authors knowledge, no prior work has brought together these ideas into an dual control and estimation analysis.

In this paper, we present a coupled non-linear estimation and control design for the attitude stabilisation of low-cost mAV vehicles. The attitude estimation algorithm is based on a the non-linear explicit complementary filter proposed in earlier work by the authors (Hamel & Mahony, 2006). This filter does not require on-line algebraic reconstruction of attitude and is ideally suited for implementation on embedded hardware platforms. Furthermore, the relative contribution of different data can be preferentially weighted in the observer response, a property that allows the designer to adjust for application specific noise characteristics. Finally, the explicit complementary filter remains well defined even if the data provided is insufficient to algebraically reconstruct the attitude, for example, for an IMU with only accelerometer and rate gyro sensors. The control algorithm considered is an adaptation of the classical passivity based control for mechanical systems (Wen & Kreutz-Delgado, 1991; Tayebi & McGilvray, 2004; Tayebi & McGilvray, 2006). We use a feed-forward control input transformation to compensate for the trajectory tracking inputs and model non-linearities. Stability is obtained using a control Lyapunov function design based on the natural mechanical passivity of rigid-body dynamics. The estimation and control analysis is undertaken in the geometric framework of the rotation group and respects all the non-linearities of rigid-body (rotational) motion. The main result is a combined attitude estimation and control algorithm. The interaction terms are bounded in terms of of estimation and control errors and the full coupled system is shown to be (almost) globally stable for at least two inertial direction measurements (ie. gravitational and magnetic fields). Simulations are provided that show the closed-loop system is well conditioned and continues to function well in the presence of significant noise and when only a single inertial direction (the gravitational field) is measured.

2 Problem definition

2.1 System Dynamics

Let $R = {}^A_BR \in \mathbb{R}^{3\times 3}$ be a rotational matrix denoting the attitude of a body-fixed frame $\{B\}$ relative to the inertial frame $\{A\}$. Let $\Omega = {}^B\Omega \in \mathbb{R}^3$ denote the angular velocity of the body-fixed frame expressed in the body-fixed frame $\{B\}$. The rigid-body dynamics of a system are given by

$$\dot{R} = R\Omega_\times \qquad \text{Kinematics} \qquad (1)$$

$$I\dot{\Omega} = -\Omega \times I\Omega + \tau \qquad \text{Dynamics,} \qquad (2)$$

where τ denotes the torque input to the dynamics and I represents the inertia matrix. We use the notation $\Omega_\times$ to denote the anti-symmetric matrix

$$\Omega_\times = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

The notation 'vex' denotes the reverse operation. Thus, $\text{vex}(\Omega_\times) = \Omega$ and $A = \text{vex}(A)_\times$ for $A = -A^T$. Let $q = (s, v)$ denote a unit quaternion (with s the scalar and v the vector component) corresponding to the rotation matrix $R = I_3 + 2sv_\times + 2v_\times^2$ and let $\mathbf{p}(\Omega) = (0, \Omega)$ denote the pure (or velocity) quaternion. The rigid-body dynamics in the quaternion representation are

$$\dot{q} = \frac{1}{2} q \otimes \mathbf{p}(\Omega) \qquad \text{Kinematics} \qquad (3)$$

$$I\dot{\Omega} = -\Omega \times I\Omega + \tau \qquad \text{Dynamics.} \qquad (4)$$

Note that the quaternion kinematics can be written explicitly in terms of the components s and v

$$\dot{v} = \frac{1}{2}\left(v_\times + sI_3\right)\Omega \qquad (5)$$

$$\dot{s} = -\frac{1}{2} v^T \Omega. \qquad (6)$$

2.2 Measurements

A typical inertial measurement unit (IMU) is equipped with 3-axis accelerometers, 3-axis rate gyroscopes and 3-axis magnetometers. Low-cost IMU systems are typically based on micro-electro-mechanical systems (MEMS) technology and have poor measurement error characteristics.

Three-axis rate gyroscopes provide measurements of the angular velocities of the body-fixed frame $\{B\}$ with respect to an inertial frame of reference $\{A\}$. The measurement is characterised by

$$\Omega_y = \Omega + b + \mu, \tag{7}$$

where Ω denotes the true value, μ denotes a (Gaussian) measurement noise process and $b := b(t)$ denotes a slowly time-varying (up to 0.05 rad/s at 0.1 Hz and lower frequencies) non-stochastic bias. The bias is typically due to a mixture of temperature effects and vibration of the IMU unit that influences the characteristics of the MEMS chip used.

Three-axis accelerometers provide measurements of the inertial acceleration of the IMU. The measurement is characterised by

$$a_y = -R^T(g_0 e_3 - \dot{v}) - \nu, \tag{8}$$

where $g_0 \approx 9.8\text{ms}^{-2}$ is the gravitational constant, $e_3 = (0, 0, 1)$, $\dot{v}$ is the acceleration of the body-fixed frame with respect to the the inertial frame and ν denotes (Gaussian) measurement noise. For the systems considered, there is no sufficiently good model of the system dynamics available to distinguish the body-fixed frame acceleration $\dot{v}$ from the gravitational component. However, for the quasi-stationary flight conditions considered for VTOL mAV systems it is reasonable to assume that the low frequency content of $\dot{v}$ is zero. We assume that there is a cutoff frequency (typically around 0.1 to 1 Hz) below which the measured acceleration $a_y \approx -g_0 R^T e_3 - \nu$ is a reasonable approximation of the gravitational force. The information used in the filter algorithm is

$$v_a = \frac{a_y}{|a_y|}, \qquad |v_a| = 1, \tag{9}$$

where $v_a \in \{A\}$ is a vector direction measurement. The estimator design is based on the assumption that v_a is a reasonable low frequency approximation of the inertial z-axis in the body-fixed frame.

The magnetometers measure the inertial magnetic field expressed in the body-fixed frame $\{B\}$

$$^{B}m = R^{T\,A}m + d + \upsilon, \tag{10}$$

where ^{A}m is the magnetic field relative to the inertial frame, d represents a bias term due to extraneous magnetic fields, and υ denotes a (Gaussian) noise term. The vectorial measurement used is

$$v_m = \frac{^{B}m}{|^{B}m|}, \qquad |v_m| = 1, \tag{11}$$

Unfortunately, many mAV systems use small scale electric motors and the bias term d in this case may dominate ($d(t) \approx \pm 2\pi/3$rad time-varying) the measurement ^{B}m making the magnetometer measurements worthless as a filter input. In the following development, the user may choose to omit the magnetometer output in the filter error term if it is deemed to have no value.

3 Combined control and estimation

In this section, we present a measurement based non-linear attitude estimation algorithm and a non-linear control algorithm for attitude control of a VTOL mAV. The main result is a coupled control Lyapunov function formulation that provides (almost) global stability of the coupled estimation/control system.

3.1 Estimation error

Let $v_i \in \mathbb{R}^3$, $i = 1, \dots, n$, denote n directional measurements. For a typical low-cost IMU the measurements are $v_1 = v_a$ (Eq. 9) and $v_2 = v_m$ (Eq. 11), or in the case v_m is unreliable, just v_1 may be used. Alternatively, directional measurements can be obtained from other sensor systems such as vision systems. For example, the direction of the sun or a normal to a horizon plane can be measured by suitable vision systems and used for navigation and stabilisation of the platform. As such we have chosen to use generic notation $\{v_i\}$ for the directional measurements, although the following development is discussed in the context of measurements obtained from an IMU.

Let v_{0i}, $i = 1, \dots, n$, denote the inertial directions of the measurements. That is, $v_{01} = v_{0a} = e_3$ is the inertial z-axis by assumption, while $v_{02} = v_{0m}$ is the inertial orientation of the earths magnetic field in the location where the experiment is undertaken.

Let $\hat{R}$ denote an estimate of the true attitude R. Let $\hat{v}_i$ denote the estimate of v_i

$$\hat{v}_i = \hat{R}^T v_{0i}. \tag{12}$$

Let $k_i > 0$, $i = 1, \dots, n$, be a sequence of positive weight gains. The estimation error is defined to be

$$\begin{aligned} E &= \sum_{i=1}^{n} k_i \cos(\angle v_i, \hat{v}_i) \\ &= \sum_{i=1}^{n} k_i v_i^T \hat{v}_i = \sum_{i=1}^{n} k_i v_{0i}^T R \hat{R}^T v_{0i} \\ &= \sum_{i=1}^{n} k_i \mathrm{tr}\left(\hat{R}^T R R^T v_{0i} v_{0i}^T R\right) = \mathrm{tr}\left(\tilde{R} P\right) \end{aligned}$$

where $\tilde{R} = \hat{R}^T R$ is the total estimation error on $SO(3)$ and

$$P = R^T \left(\sum_{i=1}^{n} k_i v_{0i} v_{0i}^T \right) R. \tag{13}$$

The matrix $P > 0$ is positive semi-definite (for $n < 3$) time-varying matrix. The weight gains $\{k_i\}$ allow the designer to weight the relative importance of different data in the filter response. Thus, if v_1 is considered to be more reliable than v_2 one would choose $k_1 > k_2$.

3.2 Estimation algorithm

The estimation algorithm proposed is the *explicit complementary filter* dynamics on SO(3) (Hamel & Mahony, 2006)

$$\dot{\hat{R}} = \hat{R}(\Omega_y - \hat{b} + k_P^o \omega)_\times, \tag{14a}$$

$$\omega = \sum_{i=1}^{n} k_i v_i \times \hat{v}_i, \tag{14b}$$

$$\dot{\hat{b}} = -k_I^o \omega, \tag{14c}$$

where $k_P^o > 0$ and $k_I^o > 0$ are constant proportional and integral observer gains, respectively. The term ω is an innovation or error term in the filter dynamics.

It is convenient to rewrite ω as an expression that recaptures the error matrix $\tilde{R} = \hat{R}^T R$. For any $u, v \in \mathbb{R}^3$, one has

$$(u \times v)_\times = (u_\times v_\times - v_\times u_\times) = v u^T - u v^T.$$

Thus,

$$\begin{aligned} \omega_\times &= \sum_{i=1}^{n} k_i (\hat{v}_i v_i^T - v_i \hat{v}_i^T) \\ &= \sum_{i=1}^{n} k_i (\hat{R}^T R R^T v_{0i} v_{0i}^T R - R^T v_{0i} v_{0i}^T R R^T \hat{R}) \\ &= \tilde{R} P - P \tilde{R}^T = 2\mathbb{P}_a(\tilde{R} P) \end{aligned}$$

where $\mathbb{P}_a(A) = (A - A^T)/2$ is the anti-symmetric projection of the matrix A.

3.3 Control error

Let $(R_d(t), \Omega^d(t))$ denote the desired attitude trajectory and let $\{D\}$ denote the desired frame of reference associated with R_d. We assume $\Omega^d \in \{D\}$ is specified in the desired frame of reference such that we have desired kinematics

$$\dot{R}_d = R_d \Omega^d_\times. \tag{15}$$

The kinematic control error is defined to be

$$\bar{R} = R_d^T \hat{R}. \tag{16}$$

Note that we use the filtered estimate $\hat{R}$ of the true attitude R to define the control error. This ensures that the error term $\bar{R}$ can be computed and used in the control design. The controller is not, however, a certainty equivalence controller, as we will provide a full coupled stability analysis. Recalling Eq. 1 along with Eqn's 15 and 16 the derivative of $\bar{R}$ yields

$$\dot{\bar{R}} = \bar{R}(\Omega_y - \hat{b} + \omega)_\times - \Omega^d_\times \bar{R}. \tag{17}$$

where ω is given by Eq. 14b. The error term associated with the system dynamics is defined to be

$$\varepsilon = \Omega - \Omega^d + k_P^v \mathrm{vex}(\mathbb{P}_a(\bar{R})), \tag{18}$$

where $k_P^v > 0$ denotes a constant proportional gain. The gain k_P^v acts like a control gain for a virtual error in the paradigm of back-stepping control design. In practice, the true angular velocity Ω is not measured and an estimate $\Omega \approx \Omega_y - \hat{b}$ is used

$$\varepsilon \approx \Omega_y - \hat{b} - \Omega^d + k_P^v \mathrm{vex}(\mathbb{P}_a(\bar{R})), \tag{19}$$

When $\hat{b}$ has converged and provides a good estimate of the gyrometer bias this ensures a reasonable (if noisy) estimate of ε. During the initial transient the dynamics of $\hat{b}$ should be compensated in the control design. In the following development we compensate for the dynamics of $\hat{b}$ in the feed-forward control transformation to ensure that the modelled error dynamics are accurate (Eq. 22). However, we use the true value of ε in the control analysis without considering the transient evolution of $\dot{\hat{b}}$. We use the approximate value of ε in the control implementation. Since the signal error occurs in the control input, it acts as a load disturbance and will be suppressed by feedback control.

The approach taken is to apply a control input transformation that contains the feed-forward terms required to track the trajectory $(R_d(t), \Omega_d(t))$ as well as compensating for non-linear transient terms due to mismatch of the filtered trajectory to the desired trajectory. The control transformation leads to error dynamics that have the classical passivity properties of rigid-body motion. The control transformation does not try to stabilise the non-linear system itself. It is a feed-forward term that ensures the subsequent stabilisation problem can be tackled as a non-linear regulation problem.

Define a non-linear control input transformation

$$\tau := \tau(\Omega, \Omega^d, R, R_d, \tau_d)$$

in terms of a new control input τ_d. The actual input τ is chosen to impose the natural mechanical structure on the error dynamics

$$I\dot{\varepsilon} = -\varepsilon \times I\Omega + \tau_d. \tag{20}$$

To determine the expression for τ one differentiates $I\varepsilon$ from Eq. 19

$$\begin{aligned} I\dot{\varepsilon} &= I\dot{\Omega} + k_I^o \omega + I(-\dot{\Omega}^d + k_P^v \mathrm{vex}(\mathbb{P}_a(\dot{\bar{R}}))) \\ &= (-\Omega \times I\Omega + \tau) + k_I^o \omega + I(-\dot{\Omega}^d + k_P^v \mathrm{vex}(\mathbb{P}_a(\dot{\bar{R}}))) \end{aligned} \tag{21}$$

Equating Eqn's 20 and 21, and solving for τ one obtains

$$\begin{aligned} \tau =& I\dot{\Omega}^d + (\Omega^d \times I\Omega) - k_P^v \mathbb{P}_a(\bar{R}) I\Omega - k_I^o \omega \\ &- k_P^v I \mathrm{vex}(\mathbb{P}_a(\dot{\bar{R}})) + \tau_d. \end{aligned} \tag{22}$$

Applying the input transformation (22) transforms the dynamics of (2) into the system

$$\dot{\bar{R}} = \bar{R}\Omega_\times - \Omega^d_\times \bar{R} \qquad \text{Error kinematics} \tag{23a}$$

$$I\dot{\varepsilon} = -\varepsilon \times I\Omega + \tau_d \qquad \text{Error dynamics,} \tag{23b}$$

where τ_d denotes the torque input to the error dynamics.

3.4 Control design

Based on the formulation of the error dynamics (23a and 23b) it is straightforward to apply standard non-linear passivity based control design technics (Wen & Kreutz-Delgado, 1991; Tayebi & McGilvray, 2004; Tayebi & McGilvray, 2006; Cha, 2006,). The control input chosen is

$$\tau_d := -k_D^c \varepsilon - k_P^c \mathrm{vex}(\mathbb{P}_a(\bar{R})), \tag{24}$$

where $k_P^c > 0$ and $k_D^c > 0$ are proportional and derivative control gains for the control. The recent work of Tayebi *et al.*(Tayebi & McGilvray, 2004; Tayebi & McGilvray, 2006) showed that the resulting closed-loop system (assuming state measurements) was almost globally asymptotically and locally exponentially stable. Note that Tayebi's results were derived in the quaternion formulation. An $SO(3)$ formulation is provided in (Cha, 2006,).

3.5 Coupled estimation and control algorithm

The main result of this paper is to prove coupled stability of the estimation and control algorithm.

We use the expression almost globally asymptotically stable to a given limit point to mean that, for almost all initial conditions, the trajectory of the systems converges to the given limit. That is, the set of initial conditions for which this does not occur are of measure zero in state space.

Lemma 3.1. *Consider the system dynamics*

$$\dot{R} = R\Omega_\times$$
$$I\dot{\Omega} = -\Omega \times I\Omega + \tau$$

with estimation dynamics

$$\dot{\hat{R}} = \hat{R}[\Omega_y - \hat{b} + k_P^o \omega]_\times$$
$$\omega = -\text{vex}\left(\sum_{i=1}^{n} k_i v_i \times \hat{v}_i\right)$$
$$\dot{\hat{b}} = -k_I^o \omega + k_I^o k_P^c \text{vex}(\mathbb{P}_a(\bar{R}))$$

and control input

$$\tau = I\dot{\Omega}^d + (\Omega^d \times I(\Omega_y - \hat{b})) - k_P^v \mathbb{P}_a(\bar{R}) I(\Omega_y - \hat{b}) - k_P^v I \text{vex}(\mathbb{P}_a(\dot{\bar{R}})) + \tau_d,$$
$$\bar{R} = R_d^T \hat{R}$$
$$\tau_d = -k_D^c \varepsilon - k_P^c \text{vex}(\mathbb{P}_a(\bar{R}))$$
$$\varepsilon = \Omega_y - \hat{b} - \Omega^d + k_P^v \text{vex}(\mathbb{P}_a(\bar{R}))$$

where $\{k_P^o, k_I^o, k_P^v, k_P^c, k_D^c\}$ *are positive gains chosen such that* $2k_P^v > k_P^c k_P^o$. *Assume that there are two or more* ($n \geq 2$) *directional measurements* v_i *available. Assume that* $\Omega(t)$ *is persistently exciting. The closed-loop system is almost globally asymptotically stable to* $\tilde{R} \to I$, $\bar{R} \to I$, $\tilde{b} \to 0$ *and* $\varepsilon \to 0$.

Proof. Consider a Lyapunov function candidate

$$V = \frac{1}{2}\varepsilon^T I \varepsilon + \frac{1}{2} k_P^c \text{tr}(I_3 - \bar{R}) + \sum_{i=1}^{n} k_I - \text{tr}(\tilde{R}P) + \frac{1}{2k_I^o}|\tilde{b}|^2, \tag{25}$$

where $\tilde{b} = b - \hat{b}$. Differentiating V, one obtains:

$$\dot{V} = \varepsilon^T I \dot{\varepsilon} - \frac{1}{2} k_P^c \text{tr}(\dot{\bar{R}}) - \text{tr}(\dot{\tilde{R}}P + \tilde{R}\dot{P}) + \frac{1}{k_I^o}\tilde{b}^T \dot{\tilde{b}}.$$

Substituting for $\dot{\tilde{b}} = -\dot{\hat{b}}$, (20), one obtains

$$\dot{V} = \varepsilon^T(-\varepsilon \times I\Omega + \tau_d) - \frac{1}{2}k_P^c \text{tr}\left(\bar{R}(\Omega_y - \hat{b} + \omega)_\times - \Omega_\times^d \bar{R}\right) - \frac{1}{2}\text{tr}(\tilde{R}P\Omega_\times - (\Omega_y - \hat{b} + \omega)_\times \tilde{R}P) + \frac{1}{k_I^o}\tilde{b}^T(\dot{b} - \dot{\hat{b}}).$$

Due to the passivity of the error dynamics $\varepsilon^T(-\varepsilon \times I\Omega) = 0$ and we assume $\dot{b}$ is a constant. Hence,

$$\dot{V} = \varepsilon^T \tau_d - \frac{1}{2}k_P^c \text{tr}\left(\bar{R}(\Omega_y - \Omega^d - \hat{b} + \omega)_\times\right) - \frac{1}{2}\text{tr}\left(\tilde{R}P(\Omega - \Omega_y + \hat{b} - \omega)_\times\right) + \frac{1}{k_I^o}\tilde{b}^T(-\dot{\hat{b}})$$

Recalling $\Omega_y \approx \Omega + b$ and substituting for $\omega = \text{vex}(\mathbb{P}_a(\tilde{R}P))$ one obtains

$$\dot{V} = \varepsilon^T \tau_d - \frac{1}{2}k_P^c \text{tr}\left(\bar{R}(\Omega + b - \Omega^d - \hat{b} + k_P^v \text{vex}(\mathbb{P}_a(\bar{R})))_\times - k_P^v \text{vex}(\mathbb{P}_a(\bar{R}))_\times + \bar{R}\omega_\times\right) - \frac{1}{2}\text{tr}(\tilde{R}P(-b + \hat{b} - \omega)_\times) - \frac{1}{2k_I^o}\text{tr}(\tilde{b}_\times^T \dot{\hat{b}}_\times)$$

Simplifying and collecting like terms one obtains

$$\dot{V} = \langle \varepsilon, \tau_d + k_P^c \text{vex}(\mathbb{P}_a(\bar{R}))\rangle - k_P^c k_P^v |\text{vex}(\mathbb{P}_a(\bar{R}))|^2 - k_P^o |\text{vex}(\mathbb{P}_a(\tilde{R}P))|^2 + k_P^c k_P^o \langle \text{vex}(\mathbb{P}_a(\bar{R})), \text{vex}(\mathbb{P}_a(\tilde{R}P))\rangle - \left\langle \tilde{b}, \left(\text{vex}(\mathbb{P}_a(\tilde{R}P)) - k_P^c \text{vex}(\mathbb{P}_a(\bar{R})) + \frac{1}{k_I^o}\dot{\hat{b}}\right)\right\rangle$$

where $\langle \cdot, \cdot \rangle$ is the vector inner product. Substituting for the control input τ_d and the adaptive bias term $\dot{\hat{b}}$ one obtains

$$\dot{V} = -k_D^c|\varepsilon|^2 - k_P^c k_P^v |\text{vex}(\mathbb{P}_a(\bar{R}))|^2 - k_P^o |\text{vex}(\mathbb{P}_a(\tilde{R}P))|^2 + k_P^c k_P^o \langle \text{vex}(\mathbb{P}_a(\bar{R})), \text{vex}(\mathbb{P}_a(\tilde{R}P))\rangle. \tag{26}$$

By completing the square, exploiting the cross term $\langle \text{vex}(\mathbb{P}_a(\bar{R})), \text{vex}(\mathbb{P}_a(\tilde{R}P))\rangle$ one can show that

$$\dot{V} = -k_D^c|\varepsilon|^2 - \frac{k_P^c k_P^v}{2}|\text{vex}(\mathbb{P}_a(\bar{R}))|^2 - k_P^o\left(1 - \frac{k_P^c k_P^o}{2k_P^v}\right)|\text{vex}(\mathbb{P}_a(\tilde{R}P))|^2 - \frac{1}{2}\left(\sqrt{k_P^c k_P^v}\text{vex}(\mathbb{P}_a(\bar{R})) - k_P^o\sqrt{\frac{k_P^c}{k_P^v}}\text{vex}(\mathbb{P}_a(\tilde{R}P))\right)^2. \tag{27}$$

Recalling the relationship $2k_P^v > k_P^c k_P^o$ and discarding the indefinite term, it follows that $\dot{V}$ is negative definite in $|\varepsilon|^2$, $|\text{vex}(\mathbb{P}_a(\bar{R}))|^2$ and $|\text{vex}(\mathbb{P}_a(\tilde{R}P))|^2$. The underlying system is smoothly defined and it is straightforward to show that all signals in the system are absolutely uniformly continuous. Consequently, Barbalat's lemma (Khalil, 1996) ensures that these error signals converge asymptotically to zero.

It follows directly that $\varepsilon \to 0$. It is straightforward to see that $\text{vex}(\mathbb{P}_a(\bar{R})) \to 0$ implies $\bar{R} \to \bar{R}^T$. The set of symmetric rotation matrices consists of the identity and the set of rotations of πrad. The unstable set πrad rotations is an unstable invariant set of the control algorithm, each such matrix representing a local maximum of the error function $\text{tr}(I - \bar{R})$. It is impossible to avoid such a set on $SO(3)$ due to its topological structure. However, due to its local maximum properties it is straightforward to see that it is locally unstable and we can say that for almost all initial conditions $R(t)$ will lead to $\bar{R} \to I$.

In Hamel *et al.*(Hamel & Mahony, 2006) it is shown that $\text{vex}(\mathbb{P}_a(\tilde{R}P)) \to 0$ implies $\tilde{R}^T = \tilde{R}$. Once this is established, nearly the same argument used above ensures that $\tilde{R} \to I$ for almost all initial conditions. In fact, some care should be taken due to influence of the integral term in the filter dynamics, details can be supplied by the authors on request. Finally, the error term $\tilde{b}$ is indefinite in the Lyapunov function derivative, however, an analysis of the invariant set properties at $\tilde{R} = I$ ensures that $\tilde{b} \to 0$. This completes the proof. □

The statement of Lemma 3.1 requires two measured directions v_1 and v_2. The filter and control can easily be implemented with only a single directional measurement (Hamel & Mahony, 2006). The authors believe that the reduced system with a single measurement v_1 will converge (for almost all initial conditions) to a limit point such that $v_1 = \hat{v}_1$ and $(R^d)^T e_3 = R^T e_3$. Furthermore, we believe that for a persistently exciting desired trajectory then all error variables would converge to zero, even in the case of a single directional measurement. These claims have not been proved and are beyond the scope of the present paper, however, simulation studies support our hypothesis.

4 Quaternion-based formulation

Using a quaternion implementation of the proposed non-linear control algorithms and filters is advantageous in the implementation of algorithms on small scale mAV with limited avionic systems. In this section, we reformulated the combined control and estimation system in terms of a quaternion formulation.

Lemma 4.1. *Consider the system kinematics, system dynamics and estimation dynamics*

$$
\begin{aligned}
\dot{q} &= \frac{1}{2} q \otimes \boldsymbol{p}(\Omega) \\
I\dot{\Omega} &= -\Omega \times I\Omega + \tau \\
\tau &= I\dot{\Omega}^d + (\Omega^d \times I(\Omega_y - \hat{b})) - k_P^v(\bar{s}\bar{v} \times I(\Omega_y - \hat{b})) \\
&\quad - k_P^v I(\dot{\bar{s}}\bar{v} + \bar{s}\dot{\bar{v}}) + \tau_d, \\
\tau_d &= -k_D^c \varepsilon_q - k_P^c \bar{s}\bar{v} \\
\varepsilon_q &= (\Omega_y - \hat{b}) - \Omega^d + k_P^v \bar{s}\bar{v} \\
\dot{\hat{q}} &= \frac{1}{2} \hat{q} \otimes \mathbf{p}(\Omega_y - \hat{b} + \omega) \\
\omega &= -\text{vex}\left(\sum_{i=1}^{n} k_i v_i \times \hat{v}_i \right) \\
\dot{\hat{b}} &= -k_I^o \psi_q + k_I^o k_P^c \bar{s}\bar{v}
\end{aligned}
$$

where $\{k_P^o, k_I^o, k_P^v, k_P^c, k_D^c\}$ are positive gains chosen such that $2k_P^v > k_P^c k_P^o$. Assume that there are two or more ($n \geq 2$) directional measurements v_i available. Assume that $\Omega(t)$ is persistently exciting. The closed-loop system is almost globally asymptotically stable to $\tilde{q} \to (1,0,0,0)$, $\bar{q} \to (1,0,0,0)$, $\tilde{b} \to 0$ and $\varepsilon_q \to 0$.

In practice, the above dual estimator/control equations can be implemented using simple Euler forward iteration and re-projection (re-normalisation) onto the set of unit quaternions. The computational advantage over expressing the estimator/control equations in their matrix form and re-projecting onto the rotation group is considerable.

5 Simulations

A set of two experiments were undertaken to demonstrate the performance of the proposed filter algorithm. All implementations of the system were undertaken for the quaternion formulation.

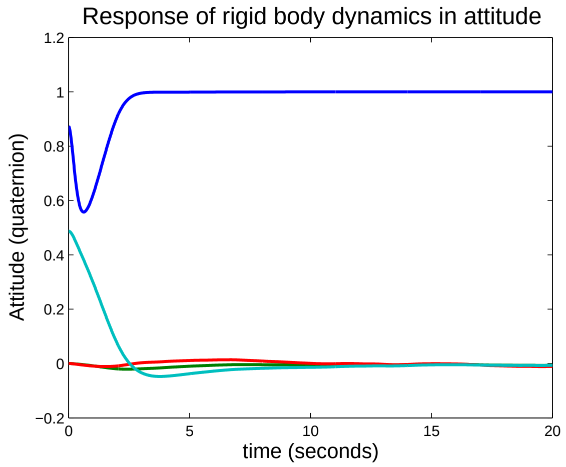

Figure 1: The evolution of the quaternion state for experiment one. The desired set point is $(1, 0, 0, 0)$. Due to the noise model, only practical convergence to a neighbourhood of the desired set point is observed.

In the first experiment, an extreme initial condition was considered to observe the transient response of the control scheme. The initial attitude of the system was set to $\pi/4$rad of roll, with 0rad pitch and yaw angles. This corresponds to the unit quaternion of $[0.8733, 0, 0, 0.4872]^T$ in the quaternion representations. The desired attitude is the stable hover position with roll, pitch and yaw all zero. This corresponds to the unit quaternion of $[1, 0, 0, 0]^T$. Gaussian noise and gyro bias terms were added to all signals associated with the IMU measurements based on measured noise characteristics of a CSIRO "EiMU" (Roberts *et al.* , 2002) IMU unit. It is assumed that the initial angular velocity is zero. The plant inertia was modelled as $0.5I_3$kg.m.s^{-2} with attitude kinematics and dynamics as discussed earlier. The controller gains were chosen to be

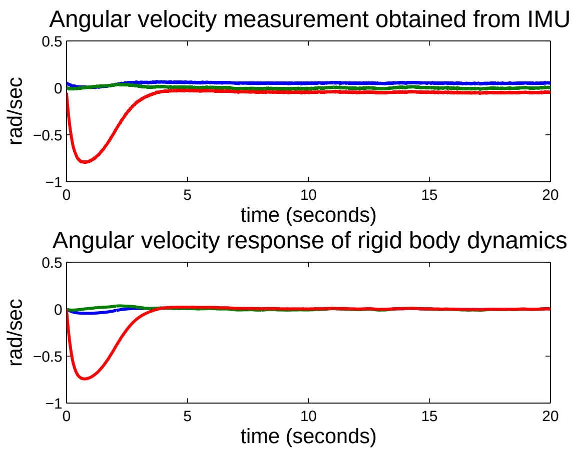

Figure 2: The evolution of the IMU measured values (top) and the true values of angular velocity for the rigid body dynamics. Note that the measured values in the top graph show a constant bias. This is compensated for the combined control design.

gain	value
k_P^o	8
k_I^o	20
k_P^v	40
k_P^c	0.02
k_D^c	2

Figure 1 shows the closed-loop trajectory for the first simulation (with full quaternion measurement). Figure 2 shows the angular velocity of the full system as well the measurement signal obtained (in simulation) from the IMU. Note the noise and bias, present in the measured signal, that is compensated for the closed-loop response. Figure 3 shows the torque demand of the control system. In this example, the transient response is convergent with time-constant around 3 seconds. The noise and bias of the measured signals do not significantly disturb the system response.

In practice, the measurement of magnetic field is often rendered useless due to magnetic fields generated by the electric motors and components of the flying vehicle. Moreover, the control inputs are subject to significant disturbances due to eddies and vor-

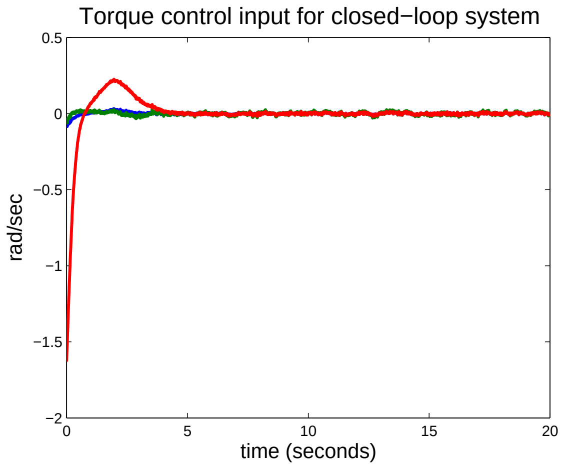

Figure 3: The simulated control input to the rigid body dynamics

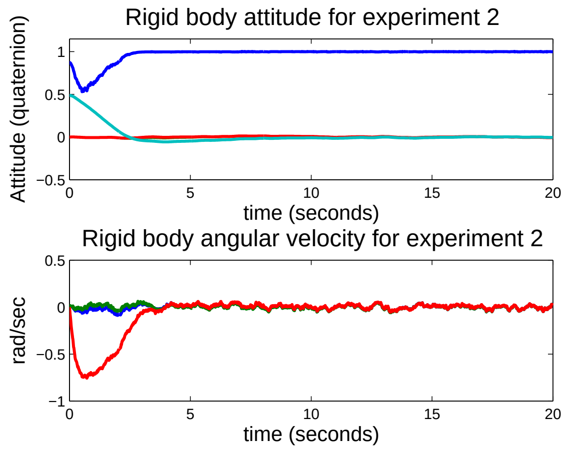

Figure 4: The simulated attitude and angular velocity using only the acceleration directional measurements

texes ingested into the rotor inflow. A second experiment was undertaken with the same initial attitude, however, only one directional measurement (gravitational direction), was used and noise was added to the control input. The additive noise model was based on the identified thrust noise model derived by Pounds *et al.*(Pounds & Mahony, 2005). Figure 4 shows the attitude and angular velocity outputs from the simulated rigid body dynamics. Note that the attitude response is qualitatively similar to that obtained in the first experiment. Only practical stability is obtained due to the actuator noise.

6 Conclusions

This paper proposes a coupled non-linear attitude estimation and control design for the attitude stabilisation of low-cost aerial robotic vehicles. The dual control is formulated to fully respect the geometric structure of the rotation group and is presented with a full control Lyapunov function analysis. A version of the control design is presented in terms of the quaternion formulation for ease of implementation on microprocessors with limited computing resources. Simulations demonstrate that the control scheme functions effectively and could be used for attitude stabilisation of mini aerial vehicles.

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