Collision-Free Workspace Design Optimisation of the 3-DOF Gantry-Tau Parallel Kinematic Machine

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Abstract

One of the main advantages of the Gantry-Tau machine is a large accessible workspace/footprint ratio compared to many other parallel machines. The Gantry-Tau improves this ratio by allowing a change of assembly mode without internal link collisions or collisions between the links and the moving TCP platform. This paper introduces an optimisation scheme based on the geometric approach for the workspace area and the functional dependencies of the elements of the static matrix and the minimum distance between two links to define the collisions between robot’s links. The results show that by careful design of the manipulated platform, a collision free workspace can be archived. Kinematic design obtained by optimisation according to this paper gives a workspace/footprint ratio of more than 3.5. In addition, a workspace optimisation method is presented where the parameters are the support frame lengths and the robot’s arm lengths.

1 Introduction

The Tau family of PKMs was invented by ABB Robotics, see Brogårdh (2002). The Gantry-Tau was designed to overcome the workspace limitations while retaining many advantages of PKMs such as low moving mass, high stiffness and no link twisting or bending moments. For a given Cartesian position of the robot each arm has two solutions for the inverse kinematics, referred to as the left- and right-handed configurations. While operating the Gantry-Tau in both left- and right-handed configurations, the workspace will be significantly larger in comparison with both a serial Gantry-type robot and other PKMs with the same footprint and with only axial forces in the links of the arms. The intended application of the robot is for machining operations requiring a workspace equal to or larger than of a typical serial-type robot, but with higher stiffness. However, the robot can also be designed for very fast material handling and assembly or for high precision processes such as laser cutting, water jet cutting and measurement.

In this paper the triangular-link variant of the 3-DOF Gantry-Tau structure is considered, which was first presented in Brogårdh, Hanssen and Hovland (2005). Triangular mounted links give several advantages: they enable a reconfiguration of the robot and a larger reach is obtained in the extremes of the workspace. When using parallel links, the orientation of the manipulated platform will be constant, which increases the risk of collisions of the arms with the manipulated platform in the extremes of the workspace area.

In Johannesson, Berbyuk and Brogårdh (2004) a basic workspace optimisation method for the 3-DOF Gantry-Tau with no triangular links was presented. Two geometrical parameters of the machine were optimised to maximise the cross-sectional workspace area. Our paper is an extension of the work in Johannesson, Berbyuk and Brogårdh (2004). The new contributions of this paper are: the optimisation is made for the Gantry-Tau with triangular mounted links, the optimisation is made over the whole workspace volume and not just the cross-sectional workspace area, the detection of the collisions between links is added to the optimisation. The geometric approach for the Gantry-Tau reachable workspace area calculation was presented in Tyapin, Hovland and Brogårdh (2007) for the first time.

Using geometrical methods the workspace can be calculated as an intersection of simple geometrical objects, Merlet (2000), for example spheres. Design optimisation was attempted by Chablat and Wenger (2003), Liu, Wang and Gao (2000) and Monsarrat and Gosselin (2003). In addition, Liu, Wang and Gao (2000) presented the relationships between the workspace and link lengths of all planar 3-DOF parallel manipulator. In Monsarrat and Gosselin (2003) the workspace was defined by three rotational angles. In our paper the static
Another interesting work is the paper by Kim, Chung and Youm (1997). In that paper a fully geometric approach to calculate the reachable workspace was presented for a 6-DOF Hexapod type PKM. The differences compared to our paper are the use of variable link lengths instead of fixed actuators at the robot base and no design optimisation was attempted in Kim, Chung and Youm (1997). The closest work to this paper was presented in Bonev and Ryu (2001) for the orientation workspace, but both Kim, Chung and Youm (1997) and Bonev and Ryu (2001) use the inverse kinematics (rotational matrix) to define the workspace.

The distance between two geometric objects (lines, segments of lines, rays, surfaces etc.) is defined as the minimum distance between two points on these objects.

The closest works to the method shown in section 3 were presented in Eberly (2001) and Teller (2008), but both use a vector cross products to define the closest distance between two line segments in 3D plane. In addition, in Eberly (2001) partitioning of the st-plane by the unit square is used, but the unit square searching algorithm has more cases and is more complicated to be implemented. Another interesting work was presented in Murray, Hovland and Brogårdh (2006). The method in Murray, Hovland and Brogårdh (2006) is based on the method proposed by Hudgens and Arai (1993) and involves calculating the minimum perpendicular distance between two finite line segments. In Murray, Hovland and Brogårdh (2006) and Hudgens and Arai (1993) vector cross products are also used but the collisions free workspace of 5-DOF the Gantry-Tau was found first. Also, no functional analysis of the distance function in 2D plane is attempted in both Eberly (2001) and Teller (2008).

The main benefit of the work in this paper is the savings in computational effort. As a result, the method presented in this paper is more accurate and faster. The time saving is possible because the conditional equations will be found from a functional dependency.

Brief descriptions of the kinematics is presented in sections 2. In section 3 the general information about a distance between two line segments is presented. The conditional equations and collisions analysis is presented in section 4. The combined optimisation problem is formulated in section 5. The results are shown in section 6 and finally the conclusions are presented in section 7.

2 Kinematic Description

In this section the kinematic description of the 3-DOF Gantry-Tau parallel kinematic machine is presented. The kinematics of the basic Gantry-Tau structure has been described earlier in Johannesson, Berbyuk and Brogårdh (2004). The difference in the current 3-DOF version of the Gantry-Tau compared to the basic version analysed in Johannesson, Berbyuk and Brogårdh (2004) is that two of the links in the three-link arm are mounted in a triangular constellation instead of in parallel. The triangular-link version of the Gantry-Tau kinematic model is illustrated in Fig.1. The 3-
Fig. 2 shows the manipulated platform. The points A, B, C, D, E and F are the link connection points. The arm with one single link connects the actuator $q_1$ with platform point F. The arm with two links connects actuator $q_2$ with the platform points A and B. The arm with three links connects actuator $q_3$ with the platform points C, D and E. Fig. 3 shows a projection of the link system in Fig. 1 into the XZ-plane and Fig. 4 a projection into the YZ-plane.

Fig. 5 shows the PKM structure in both the left-handed and right-handed configuration (also called assembly or working modes). The Tau structure is characterised by a clustering of the links in groups of 1, 2 and 3, respectively, with fixed link lengths $L_1$, $L_2$ and $L_3$. Three linear actuators are used at the base to move the three arms independently in the global X direction. More details about the inverse and forward kinematics of the Gantry-Tau can be found in Brogårdh, Hanssen and Hovland (2005) and Williams, Hovland and Brogårdh (2006).

The actuator track locations are fixed in the Y and Z directions and the locations are denoted $T_{1y}$, $T_{1z}$, $T_{2y}$, $T_{2z}$, $T_{3y}$ and $T_{3z}$, respectively (see Figs.4, 5 and 1). The dimensioning of the PKM’s support frame is given by the two variables $Q_1$ and $Q_2$ as illustrated in Fig. 5, where $Q_1$ is the depth and $Q_2$ is the height. The width of the machine in the X direction is given by the length of the actuators.

For 5-DOF the Gantry-Tau with the triangular link pair the platform points A – F are rotated in the following order: First a rotation of $r_y = \alpha$ about the platform Y axis which initially coincides with the global Y axis. Second a rotation $r_z$ about the global Z axis and third a rotation $r_x$ about the global X axis. The angle rotation $\alpha$ is performed first to avoid an additional transforma-
tion from the platform Y-axis to the global Y-axis. The platform points A–F in the TCP coordinate frame are calculated as follows.

\[
[a_x \ a_y \ a_z]^T = R_y(0) \begin{bmatrix} 0 \ -L_{tool} \ (R_p + L_{pin} + \frac{L_b}{2}) \end{bmatrix}^T
\]

\[
[b_x \ b_y \ b_z]^T = R_y(0) \begin{bmatrix} 0 \ -L_{tool} - L_p \ (R_p + L_{pin} + \frac{L_b}{2}) \end{bmatrix}^T
\]

\[
c_x \ c_y \ c_z]^T = R_y \begin{bmatrix} \frac{\pi}{3} \ 0 \ -L_{tool} \ (R_p + L_{pin} + \frac{L_b}{2}) \end{bmatrix}^T
\]

\[
d_x \ d_y \ d_z]^T = [c_x \ b_y \ c_z]^T
\]

\[
e_x \ e_y \ e_z]^T = R_y \begin{bmatrix} \frac{\pi}{3} \ 0 \ (R_r + L_{pin} + \frac{L_b}{2}) \end{bmatrix}^T + \cdots + R_y(0) \begin{bmatrix} 0 \ -L_{tool} - \frac{L_b}{2} \ 0 \end{bmatrix}^T
\]

\[
f_x \ f_y \ f_z]^T = [e_x \ b_y \ e_z]^T
\]

The vectors pointing from the actuator positions to the points A, B, C, D, E, F on the platform are given below.

\[
A = [a_x \ a_y \ a_z]^T
\]

\[
B = [b_x \ b_y \ b_z]^T
\]

\[
C = [c_x \ c_y \ c_z]^T
\]

\[
D = [d_x \ d_y \ d_z]^T
\]

\[
E = [e_x \ e_y \ e_z]^T
\]

\[
F = [f_x \ f_y \ f_z]^T
\]

The algorithm to define the closest distance between two TCPs is given below.

\[
X_i \leq X \quad 0^0 \leq r_y \leq 90^0
\]

while the constraints for the right-handed configuration are

\[
X_i \geq X \quad -90^0 \leq r_y \leq 0^0
\]

The \( \cos \alpha \) and \( \sin \alpha \) equations are given below:

\[
\cos \alpha = \frac{T_{3z} - Z}{\sqrt{L_{3m}^2 - (Y + M_y - T_{3y})^2 + M_x^2 + M_z^2}}
\]

\[
\sin \alpha = \sqrt{1 - \cos^2 \alpha}
\]

\( L_{3m} \) is the middle length of the triangular-mounted arm. \( M_x, M_y, M_z \) are coordinates of a vector from a midpoint \( M \) between the triangular link coordinates \( C \) and \( E \) on the platform to the actuator position \( T_{3z}T_{3y}T_{3z} \), see Fig. 2 and given below:

\[
M_x = C_x + \frac{E_x - C_x}{2}
\]

\[
M_y = E_y
\]

\[
M_z = C_z + \frac{E_z - C_z}{2}
\]

A prototype of the 3-DOF Gantry-Tau with a triangular-mounted link pair built at the University of Agder, Norway is shown in Fig. 6. The kinematic parameters of the prototype are given below.

\[
Y_{offs} = 0.125
\]

\[
Z_{offs} = 0 \quad Q_1 = 0.5 \ m \quad Q_2 = 1 \ m
\]

\[
T_{1y} = -Q_1 \quad T_{1z} = Q_1 \quad T_{2y} = 0
\]

\[
T_{3z} = Q_2 \quad T_{3y} = 0 \quad T_{3z} = 0
\]

\[
T_{4} = T_{2y} + Y_{offs} \quad T_{1z} = T_{2z} - Z_{offs}
\]

\[
T_{2y} = T_{2y} - Y_{offs} \quad T_{3z} = T_{2z} - Z_{offs}
\]

\[
T_{3y} = T_{3y} + Y_{offs} \quad T_{3z} = T_{3z} + Z_{offs}
\]

\[
T_{4} = T_{3y} - Y_{offs} \quad T_{4} = T_{3z} + Z_{offs}
\]

\[
T_{5y} = T_{3y} \quad T_{5z} = T_{3z} + Z_{offs}
\]

\[
T_{6y} = T_{1y} + Z_{offs} \quad T_{6z} = T_{1z} - Y_{offs}
\]

where \( Y_{offs} \) and \( Z_{offs} \) are distances from the base plate to the universal joint in Y and Z axis, \( T_{1y} \) \( T_{1z} \) are arm actuator positions and \( T_{1y}^i \) \( T_{1z}^i \) are link actuator positions.

3 The Distance Between Two Segments

The algorithm to define the closest distance between two line segments is presented. The distance between two
geometric objects (lines, segments of lines, rays, surfaces etc.) is defined as the minimum distance between two points on these objects. Two lines \( P(s) \) and \( Q(t) \) are shown in Fig.7 and parametric equations are given below.

\[
P(s) = P_0 + s(P_1 - P_0) = P_0 + su \\
Q(t) = Q_0 + t(Q_1 - Q_0) = Q_0 + tv
\]  

(21)

(22)

where \( P_0 \) and \( Q_0 \) are start points of line segments and \( P_1, Q_1 \) are end points, \( u \) and \( v \) are vectors pointed between start and end point of the line segments.

A vector between two points on line segments is given below.

\[
w(s,t) = P(s) - Q(t)
\]

(23)

where \( P(s) \) and \( Q(t) \) are limited by line segment lower and upper boundaries. The distance between points \( P(s_c) \) and \( Q(t_c) \) on the line segments is a length of a vector \( w(s_c,t_c) \) and equals a minimum distance between two lines. According to the equations eqs.(21-23), the vector \( w_c \) is given below.

\[
w_c = P(s_c) - Q(t_c) = P_0 + s_c u - Q_0 - t_c v = w_0 + s_c u - t_c v
\]

(24)

where the vector \( w_0 \) is pointed between the line segment start points.

In addition, the vector \( w_c = w(s_c,t_c) \) is also perpendicular to both unit vectors \( u \) and \( v \).

\[
u \cdot w_c = 0
\]

(25)

\[
v \cdot w_c = 0
\]

(26)

According to the eqs.(24-25), the vector scalar products \( u \cdot w_c \) and \( v \cdot w_c \) are given below.

\[
(u \cdot u)s_c - (u \cdot v)t_c = -u \cdot w_0
\]

(27)

\[
(v \cdot u)s_c - (v \cdot v)t_c = -v \cdot w_0
\]

(28)
The length of the vector \( \mathbf{w} \) calculations are necessary. If the points are located outside infinite lines as given in eq.

\[
s_c = \frac{b'e' - c'd'}{a'e' - b'^2} \quad t_c = \frac{a'e' - b'd'}{a'e' - b'^2}
\]

(29)

where \( a', b', c', d', e' \) are help variables

\[
a' = \mathbf{u} \star \mathbf{u} \quad b' = \mathbf{u} \star \mathbf{v} \quad c' = \mathbf{v} \star \mathbf{v} \quad d' = \mathbf{u} \star \mathbf{w}_0 \quad e' = \mathbf{v} \star \mathbf{w}_0
\]

(30)

The denominator of the parameters \( s_c \) and \( t_c \) is greater or equals zero and given below.

\[a'c' - b'^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (|\mathbf{u}| |\mathbf{v}| \cos \gamma)^2 = (|\mathbf{u}| |\mathbf{v}| \sin \gamma)^2 \]

(31)

where \( \gamma \) is an angle between vectors \( \mathbf{u} \) and \( \mathbf{v} \). If the denominator is greater than zero, the lines are not parallel. If the denominator equals zero, the lines are parallel.

A distance between line segments and a distance between their extended lines may be different because the closest points between infinite line may be located outside of the segment range. The line segments between points \( [P_0; P_1] \) and \( [Q_0; Q_1] \) are given by eqs.(21-22), but two conditions are applied.

\[0 \leq s \leq 1 \quad 0 \leq t \leq 1 \quad (32)\]

The closest distance between two line segments will be found in four stages.

Stage 1 : Check the segments if they are parallel. The parameters \( s_c \) and \( t_c \) for the parallel segments will be found from eq.(29), where one parameter equals zero. For example, \( s_c = 0 \), the solution of the equation system is given below.

\[s_c = 0 \quad \Rightarrow \quad s_c = \frac{b'e' - c'd'}{a'e' - b'^2} = 0 \quad \Rightarrow \]

\[b'e' - c'd' = 0 \quad \Rightarrow \quad b'e' = c'd' \quad \Rightarrow \]

\[c' = \frac{c'd'}{b'} \quad \Rightarrow \quad t_c = \frac{a'e' - b'd'}{a'e' - b'^2} \quad \Rightarrow \]

\[t_c = \frac{a'c'd' - b'd'}{a'e' - b'^2} \quad \Rightarrow \quad t_c = \frac{a'c'd' - b^2d'}{b(a'e' - b'^2)} \quad \Rightarrow \]

\[t_c = \frac{d(a'c' - b^2)}{b'(a'e' - b'^2)} \quad \Rightarrow \quad t_c = \frac{d'}{b'} = \frac{e'}{\bar{c'}}
\]

The parameters are \( s_c = 0 \) and \( t_c = \frac{e'}{\bar{c'}} = \frac{d'}{b'} \), if segments are parallel and stage 3 is applied to check the vertexes of the lines.

Stage 2 : Calculate the parameters \( s_c \) and \( t_c \) for the infinite lines as given in eq.(29). If the closest points are located inside of the segment ranges, no additional calculations are necessary. If the points are located outside of the segment ranges, new points will be found to define the length of the vector \( \mathbf{w}_c \) as a minimum distance for the given segments and the third stage is applied.

Stage 3 : The boundary search method is used in this stage. A minimisation of the vector \( \mathbf{w} \) is the same as a minimisation of \( \mathbf{w}_c^2 \).

\[\mathbf{w}_c^2 = (\mathbf{w}_0 + s\mathbf{u} - t\mathbf{v})(\mathbf{w}_0 + s\mathbf{u} - t\mathbf{v})\]

where \( \mathbf{w}_c^2 \) is a parabolic function of \( s \) and \( t \), and a parabola will be defined in the \( st \) plane, where the minimum of the parabolic function is located at the point \( C(s_c, t_c) \).

In Fig.8 the minimum of the parabolic function \( C(s_c, t_c) \) and the unit square \( Region_0 \) are shown. The minimum \( C(s_c, t_c) \) is located outside of the region \( Region_0 \), because the closest points between two line segments are located outside of the segment ranges. In this stage the "visible" boundaries for the point \( C(s_c, t_c) \) will be found.

In Fig. 8 four boundaries of the unit square \( Region_0 \) are given by \( s = 0, s = 1, t = 0, \) and \( t = 1 \). The point \( C(s_c, t_c) \) is located in \( Region_2 \) in Fig.9. The boundary \( s = 0 \) is "visible" for the point \( C(s_c, t_c) \) if \( s_c < 0 \). The boundary \( s = 1 \) is "visible" for the point \( C(s_c, t_c) \) if \( s_c > 1 \). The boundary \( t = 0 \) is "visible" for the point \( C(s_c, t_c) \) if \( t_c < 0 \). The boundary \( t = 1 \) is "visible" for the point \( C(s_c, t_c) \) if \( t_c > 1 \). Up to two boundaries are "visible" for the point \( C(s_c, t_c) \) if the point is located inside of diagonal regions \( Region_2, Region_4, Region_6, Region_8 \) and one boundary if the point is located inside of \( Region_1, Region_3, Region_5, Region_7 \).

In addition, a position of the minimum is found for each boundary. For example, the boundary is \( s = 0 \), \( \mathbf{w}_c^2 = (\mathbf{w}_0 + 0\mathbf{u} - t\mathbf{v})^2 = (\mathbf{w}_0 - t_c\mathbf{v})^2 \). The minimum of \( \mathbf{w}_c^2 \) is found from derivation.

\[0 = \frac{d}{dt}\mathbf{w}_c^2 = -2\mathbf{v}(\mathbf{w}_0 - t_c\mathbf{v}) \quad t_c = \frac{\mathbf{v} \star \mathbf{w}_0}{\mathbf{v} \star \mathbf{v}} = \frac{e'}{\bar{c'}}
\]

(33)

The minimum is located at the point \( (0; t_c) \) if \( t_c \) is inside of the range \( 0 \leq t_c \leq 1 \). The minimum is located at the limits of the boundary \( t_c \) if \( t_c \) is outside of the range \( 0 \leq t_c \leq 1 \) and two possible locations of the minimum are \((0; 0)\) or \((0; 1)\). Other three boundaries will be found in the similar manner.

However, two boundaries are "visible" for the minimum \( C(s_c; t_c) \) in regions 2, 4, 6, 8. Both boundaries will be taken into account and five possible solutions will be checked (two solutions for each boundary and one solution at the common point). In this stage the computational time will be reduced by the use of both boundaries at one time. The diagram of the boundary search method is shown in Fig.10.

Stage 4. Calculate the closest distance between two points as a length of the vector \( \mathbf{w}_c \) from eq.(24).
4 Collisions analysis

This section presents new results compared to previous papers about the Gantry-Tau machine. The purpose of this work is to find a new method to quickly check for internal link collisions. A collision is deemed to have occurred if the minimum distance between two links in a given orientation is less than the diameter of the links. The conditions $S_0 - T_{24}$ for fifteen pairs of the links are given without identification and explanation how they are relevant because of limited space. The conditions were tested with different arm lengths and parameters $Q_1$ and $Q_2$. Full conditional equations and the explanations are given in Tyapin (2008).

Pairs $C - D$ and $A - B$ are parallel. The condition $S_0$ (see Fig.10) is used to define the minimum distance between the links. Additional check of the segment vertices gave the negative result and conditions $T_{11} - T_{24}$ (see Fig.10) are not applied for the Gantry-Tau parallel links. The coordinates of the minimum $C(s_c, t_c)$ are $s_c = 0$ and $t_c = \frac{c}{e}$ and a vector $w_c$ is given below.

$$w_c = w_0 - \frac{e'}{e''}v$$  \hspace{1cm} (34)

According to the section 2 and eq.(34), the minimum distance between links $Dis = |w_c|$ is given as follow.

$$Dis_{AB} = \frac{2T'_{iy}}{L_i} \sqrt{L_i^2 - (i_y + Y - T'_{ky})^2}$$  \hspace{1cm} (35)

where $T'_{iy}$ is the actuator position of the links $D$ or $B$, $T'_{ky}$ is the actuator position of the links $A$ or $C$, $i_y$ is $Y$-coordinate of the points $D$ or $B$, $L_i$ is a length of the vectors $D$ or $B$. All these parameters are constants and $Y$ is a current $Y$-coordinate of the TCP.

The minimum distance equation for the links $C - D$ or $A - B$ depends on $Y$-position of the TCP and the distance will be found once for fixed $Z$ and unconditioned $Y$-coordinate of the TCP.

Pairs $E - A$ and $E - B$ are not parallel, and the condition $S_3$ is applied for any $YZ$-position of the TCP, see Fig.10. The coordinates of the minimum $C(s_c, t_c)$ are $s_c = 1$ and $t_c = \frac{c' + b'}{e'}$. A vector $w_c$ for the pair $E - A$ is given below.

$$w_c = E_A - \frac{e' + b'}{e''}v$$  \hspace{1cm} (36)

where a vector $E_A$ is pointed from the actuator position of the link $A$ to the point $E$ on the platform.

A parameter $t_c$ will be simplified and an equation is given below.

$$t_c = \frac{1}{L_A} \left( dX^2 + dY^2 + e_x a_x + a_z e_z + a_y e_y + ... + dY(e_y) + \cos(a_x dX + a_z dZ + e_z dZ) + ... + dY(a_y) + \sin(a_x dX + e_z dX - a_z dZ - e_x dZ) \right)$$  \hspace{1cm} (37)

Note that a vector $w_c$ is the same for a pair $E - B$, but instead of $(a_x, a_y, a_z)$ the coordinates of the point $E$ are used.

According to the section 2 and eq.(36), the minimum distance between links $Dis = |w_c|$ is given as follow.

$$Dis_{EA} = \sqrt{|E_A - t_c A|}$$  \hspace{1cm} (38)

The minimum distance equation for the links $E - A$ or $E - B$ depends on $Y$ and $Z$-position of the TCP. The distance will be found for each point of the workspace, but the constant coordinates of the points on the platform in TCP coordinate frame will be added into account.

Pairs $E - C$ and $E - D$ are not parallel, and the conditions $S_2$ and $T_{14}$ (see Fig.10) are applied and depend on $Y$-position of the TCP. The coordinates of the minimum $C(s_c, t_c)$ are $s_c = 0$ and $t_c = \frac{c}{e}$ (condition $S_2$) and $s_c = \frac{a - d}{a'}$ and $t_c = 0$ (condition $T_{14}$).

The condition $S_2$ is used if the $Y$-coordinate of the vector $C$ or $D$ is negative. Since the $Y$ coordinate is positive the condition $T_{14}$ is applied. The $Y$ coordinate is a control point (see section 3). According to the section 2 the minimum distance between links will be found as follows.

$$if \quad (d_y + dY_4) < 0 \quad or \quad (c_y + dY_3) < 0 \quad (39)$$

$$1 : \quad Dis_{EC} = \frac{T'_{ky}}{L_k} \sqrt{L_k^2 - (k_y + Y - T'_{ky})^2}$$  \hspace{1cm} (40)

$$if \quad (d_y + dY_4) \geq 0 \quad or \quad (c_y + dY_3) \geq 0 \quad (41)$$

$$2 : \quad Dis_{EC} = \frac{T'_{iy}}{L_i} \sqrt{L_i^2 - (i_y + Y - T'_{iy})^2}$$  \hspace{1cm} (42)

where $i$ indicates the vector $E$ and $k$ indicates the vector $C$ or $D$, where $T'_{ky}, T'_{iy}, k_y, i_y, L_i, L_k$ are constants and $Y$ is current $Y$-coordinate of the TCP. The minimum distance function for the links $E - C$ and $E - D$ only depends on $y$-position of the TCP.

Pairs $C - A$ and $C - B$ are not parallel and conditions $S_3$, $T_{23}$ (see Fig.10) are applied. The coordinates of the minimum $C(s_c, t_c)$ are $s_c = 1$ and $t_c = \frac{c' + b'}{e'}$ (condition $S_3$) and $s_c = 1$ and $t_c = 1$ (condition $T_{23}$).

The distance equation for the pair $C - A$ is not the same as for $E - A$ but will be simplified in the similar way. The general equation for the condition $S_3$ is given
as follows.

\[ D_{is3} = \sqrt{(U_x - T'_{ux})^2 + (V_x + T''_{ux})^2 + ... + (U_y - T'_{uy})^2 + (V_y + T''_{uy})^2 + ... + (U_z - T'_{uz})^2 + (V_z + T''_{uz})^2} \]

where \( U_x, U_y, U_z \) are the coordinates of a first vector and \( V_x, V_y, V_z \) are the coordinates of a second vector and \( K_{17}(x, y, z) \) is given below.

\[ K_{17}(x, y, z) = \frac{1}{L_U^T}([2U_x - T'_{ux}]^2 + ... + [2U_y - T'_{uy}]^2 + ... + [2U_z - T'_{uz}]^2) \]

In the next stage the kinematics will be used to simplify the equation 43. In addition, the control points will be found as given below.

\[ s_N > s_D \Rightarrow b'(e' + b') > c'(a' + d') \]  

where \( a' + d' = U_x - T'_{ux} \) and \( b' + e' = K_{17}(x, y, z)L_U^T \)  

\[ \Rightarrow (U_y - T'_{uy})^2 + (U_z - T'_{uz})^2 > (U_x - T'_{ux})^2 \]  

The inequality 44 is easy to check if all kinematic parameters are used instead of variables.

The general distance equation for the condition \( T_{23} \) is given below.

\[ D_{is23} = \sqrt{(u_x - u_x - T'_{ux})^2 + (u_y - u_y - T'_{uy})^2 + (u_z - u_z - T'_{uz})^2} \]

where \( u_x, u_y, u_z \) are the coordinates of the first vector.

5 Optimisation problem

An optimal design for the Gantry-Tau (and other PKMs) is difficult to find manually. In this section an optimisation scheme based on the geometric descriptions of the workspace, unreachable area and a functional dependency of the collisions calculation is presented. The optimisation problem is expressed as

\[ \max V_R(Q_1, Q_2, L_1, L_2, L_3) \]

subject to

\[ V_U(Q_1, Q_2, L_1, L_2, L_3) \]

where \( V_R \) is the total workspace volume, \( L_U^C \) is the minimum distance between two robot’s links. \( V_U \) is the unreachable area volume in the middle of the workspace caused by long arms and short actuators. These area exist even if the robot can be reconfigured between the left-hand and right-hand inverse kinematic solutions and can significantly reduce the workspace of the Gantry-Tau. The unreachable area was presented before in Tyapin,
Hovland and Brogårdh (2007a). The collisions are detected if the distance is less or equals 0.05 m (the diameter of the link). \( L_A - L_F \) are the link lengths. \( Q_1, Q_2 \) are the support frame parameters.

The total workspace volume \( V_R \) as a function of the two support frame design parameters \( Q_1, Q_2 \) and the individual link lengths \( L_A - L_F \) is maximised while the minimum distance between the links is greater than 0.05 m (no collisions are detected) and the unreachable area volume equal to zero. Since the track lengths are fixed and equal to 2.0 m, the unreachable area volume will appear if the link lengths become too large. Hence, the unreachable volume is effectively an upper bound on the total achievable workspace when fixed length actuators are used. Without including the unreachable area volume into the workspace optimisation, it would not be possible to simultaneously optimise both the support frame parameters \( Q_1, Q_2 \) and the link lengths \( L_A - L_F \) as these would all go to infinity. The optimisation results are presented in section 6.

6 Results

The final optimisation design parameters of the Gantry-Tau were found using the *lsqnonlin* function in Matlab. The optimisation results are given below:

\[
\begin{align*}
Q_1 &= 0.5155 & Q_2 &= 1.0212 & V_R &= 3.6999 \\
L_C^* &= 0.0901 & V_U &= 0 \\
L_F &= 0.9482 & L_A &= 0.9514 & L_E &= 0.9467 \\
L_C &= 0.9467 & L_B &= 0.9514 & L_D &= 0.9467
\end{align*}
\]

(54)

These results would have been difficult to obtain by a manual design, as all the link lengths are different and \( Q_2 \) is different from 2\( Q_1 \) which has been a typical manual design choice of the Gantry-Tau in the past. The required installation space of the Gantry-Tau equals 2\( Q_1 Q_2 = 1.05 \text{m}^3 \) for the optimised design. Hence, the total workspace to installation space ratio for the optimised design is \( V_{installation} = 3.5228 \text{ m}^3 \) which is large compared to most other PKMs which typically have a ratio of less than one.

7 Conclusions

A new design optimisation of 3-axis version of the Gantry-Tau parallel kinematic manipulator has been presented in this paper. In addition, the geometric approach to define the workspace and the functional dependency of the cross-sectional workspace area on the robot’s \( X \) coordinate is used to calculate the total workspace volume (see Tyapin, Hovland and Brogårdh (2007)) and the unreachable area volume (see Tyapin, Hovland and Brogårdh (2007a)) and the functional dependency to detect the collisions between links have been developed. The collision detection approach is an extension of the method in Murray, Hovland and Brogård (2006). The method is based on a functional dependency of the elements of the static matrix \( \mathbf{H} \) and the Cartesian position vector \( \mathbf{X} \). In addition, the relations between vectors were taken into account.

The Gantry-Tau is a part of HEXAPOD family parallel kinematic manipulator and the use of the conditional equations to define the collisions free workspace is applicable for the other machines from the family (H4, Orthoglide, Delta, etc.), but an additional analysis of the links properties is necessary.

<table>
<thead>
<tr>
<th>Method</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Method in Murray (2006)</td>
<td>15</td>
</tr>
<tr>
<td>Method in Teller (2008)</td>
<td>11</td>
</tr>
<tr>
<td>Functional Dependency</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1: Collisions detection computation time for three different methods.

Table 1 shows the computational requirements for the three different approaches on the triangular version of the 3-DOF Gantry-Tau PKM. The method based on the functional dependency is 15 times faster than the method presented in Murray, Hovland and Brogård (2006). The computational time has been normalized to 1 for the third approach. The drawbacks of the method presented in this paper are conditional control points. For example, the conditions \( S_3 \) and \( T_{24} \) (see Fig.10) are used to define the closest distance between two line segments, where the condition \( S_3 \) is applicable while the \( Y \) coordinates of the segment end points are positive. The additional calculation is used to find the \( Y \) coordinates, when the condition \( S_3 \) changes to \( T_{24} \).

The collision analysis shows that the design of the manipulated platform is crucial to avoid link collisions and different platform design could be needed if the robot will be used in both left hand and right hand configurations. An optimisation routine for the platform design would be a challenging future research topic. In addition, the algorithms developed in this paper allow for a fast workspace analysis and customisation of each individual Gantry-Tau machine design, depending on the work object requirements. For a complete automation design, potential collisions with the work objects should also be considered.

The results in this paper show that it is possible to optimise the kinematic design of the Gantry-Tau PKM to achieve no collisions between links while maximising the reachable workspace and keeping unreachable area equals zero.
Future extensions of the work presented in this paper will introduce performance criteria such as the Cartesian stiffness, singularities and first resonance frequency into the design optimisation.

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